Central European University

Department of Mathematics and its Application

Heegaard Floer Invariants of Smooth and Legendrian Links in Rational Homology 3-spheres

by

Alberto Cavallo

Supervisor

Prof. András I. Stipsicz

BUDAPEST, HUNGARY

2018

Declaration

I hereby declare that this dissertation contains no materials accepted for any other degrees in any other institutions.

Moreover, the dissertation contains no materials previously written and/or published by another person, except where appropriate acknowledgment is made in the form of bibliographical reference.

Budapest, April 10 2018 Alberto Cavallo

Abstract

Our main goal in this thesis is to describe the most important results on smooth and Legendrian knots obtained from Heegaard Floer homology in the last ten years and generalize some of them to links.

Our first result is a link version of the Thurston-Bennequin inequality. This inequality, in its usual formulation, gives an upper bound for the maximal Thurston-Bennequin and self-linking numbers of a knot K in a tight contact 3-manifolds (M, ξ), in terms of the Euler characteristic of a Seifert surface for K. We generalize the inequality to every link L, where the resulting upper bound involves the Thurston norm of L. This is a rational number extracted from the semi-norm, introduced by Thurston, on the relative second homology group of a 3-manifold with toric boundary.

We say that two n-component links in S^3 are strongly concordant if there is a cobordism between them consisting of n disjoint annuli, each one realizing a knot concordance. Then, starting from a grid diagram D of a link L, we define a filtered chain complex $\left(\widehat{GC}(D), \widehat{\partial}\right)$ and we prove that its homology, denoted with $\widehat{\mathcal{HFL}}(L)$, is a strong concordance invariant.

Since we can prove that $\widehat{\mathcal{HFL}}(L)$ has dimension one in Maslov grading zero, we also extract a numerical invariant from the homology group that we call $\tau(L)$. The τ-invariant gives a lower bound for the slice genus $g_4(L)$, which is the minimum genus of an oriented, compact surface properly embedded in D^4 and whose boundary is L. We also show that there is a strict relation between the filtration levels of $\widehat{\mathcal{HFL}}(L)$ and the Alexander grading of the torsion-free quotient of $cHFL^-(L)$, a different bigraded version of link Floer homology.

Furthermore, we define an invariant of Legendrian links by using open book decompositions. This is done by describing a suitable condition for an open book decomposition (B, π, A) to be adapted to a Legendrian link L, where A is a system of generators of the first relative homology group of $\overline{\pi^{-1}(1)}$. At this point, we define a special Heegaard diagram D for the link L in the 3-manifold $-M$, given by reversing the orientation on M; we have that D is obtained up to isotopy from (B, π, A) and then, when L is zero in homology, the invariant $\mathfrak{L}(L, M, \xi)$ is the isomorphism class of a distinguished cycle $\mathfrak{L}(D)$ in the link Floer complex $cCFL^-(D, \mathfrak{t}_\xi)$.

We also give some results on quasi-positive links in S^3. In particular, we introduce the subfamily of connected transverse $\mathbb{C}$-links: these are links such that the surface Σ_B, associated to a quasi-positive braid B for L, is connected. We show that, for this kind of links, the slice genus g_4 is determined by τ. This allows us to prove that the slice genus is additive under connected sums of connected transverse $\mathbb{C}$-links.

Contents

List of Figures

(...) while a cloud of smoke settled heavily over the battlements in the distinct colossal figure of-a horse.

E. A. Poe

Introduction

In the early years of the 21st century, Ozsváth and Szabó introduced an invariant [68], that they decided to call Heegaard Floer homology, which had a great impact in the study of low dimensional topology. This is an invariant of 3-manifolds up to diffeomorphism; it is obtained from a particular presentation of the 3-manifold M, called a Heegaard diagram, consisting of a closed, oriented, genus g surface Σ, together with two sets α and β of g independent, disjoint, simple closed curves and a basepoint w in Σ.

A chain complex over $\mathbb{F}$, the field with two elements, can be associated to a Heegaard diagram, where the generators are combinatorially defined from α and β. The differential is gotten by counting J-holomorphic disks in the g-fold symmetric power of Σ, equipped with an almost-complex structure J, as we explain in Subsection 4.1.1. The Heegaard Floer homology group is the homology of this chain complex. There exist many different versions of the homology, depending on how many J-holomorphic disks we take into account, but all of them are invariants of M. One of the most useful, that we define in Section 5.2, is denoted with $\widehat{HF}(M)$.

Heegaard Floer homology groups possess an $\mathbb{F}$-splitting induced by Spin^c structures. Using a definition of Turaev in [88], we say that a Spin^c structure on M is the homotopy class, away from a point, of a 2-plane field on M. This means that we can write a summand of $\widehat{HF}(M)$ as $\widehat{HF}(M, \mathfrak{t})$, where $\mathfrak{t}$ is a Spin^c structure on M.

Moreover, we have an additional grading, called Maslov grading, in the case when some conditions are satisfied; more specifically, we require the first Chern class of $\mathfrak{t}$ to be a torsion class. This always happens if M is a rational homology 3-sphere, which is a 3-manifold with rational homology isomorphic to the one of the 3-sphere S^3; in fact, in this case the second integer cohomology group is a finite group. For this reason, and the fact that many proofs are easier in this setting, we suppose that 3-manifolds are rational homology spheres unless the converse is explicitly written.

Ozsváth and Szabó found a way to extend Heegaard Floer homology to an invariant of links in 3-manifolds [69], in this setting called link Floer homology. This was discovered independently by Rasmussen in his doctoral thesis [78].

Moreover, in [70] they use it to define an invariant of contact 3-manifolds. A contact structure ξ on M is a cooriented 2-plane field, given as the kernel of a 1-form α on M, such that $\alpha \wedge d\alpha$ is a volume form for M. We say that ξ is overtwisted if there is an embedded disk E in M whose boundary is a Legendrian knot, which means that at each point the tangent vector is contained in the contact structure, and the contact framing of ∂E has zero twisting along E. The last sentence can be rephrased by saying that the Thurston-Bennequin number of ∂E is equals to zero, see Chapter 3. We say that the structure ξ is tight if it is not overtwisted.

In [70] the contact invariant $\widehat{c}(M, \xi)$ is defined. This is done by identifying the isomorphism class of an element in the group $\widehat{HF}(M, \mathfrak{t}_\xi)$, where $\mathfrak{t}_\xi$ is the Spin^c structure induced by ξ. The class $\widehat{c}(M,\xi)$ is a contact isotopy invariant of (M,ξ). One of its applications consists of a simpler proof of a well-known result of Eliashberg and Gromov in [25], which says that a symplectically fillable contact structure on a rational homology sphere is always tight.

A contact strucutre ξ is weakly symplectically fillable if there is a compact symplectic 4-manifold (X,ω) such that $\partial X = M$ and $\omega|_\xi > 0$. It follows from [70] that $\widehat{c}(M,\xi)$ vanishes if ξ is overtwisted, while the contact invariant is non-trivial when the contact structure is symplectically fillable.

Our main goal in this thesis is to describe, in the best way we can, the most important results on smooth and Legendrian knots obtained from Heegaard Floer homology in the last ten years and generalize some of them to links.

After giving some background on smooth links in 3-manifolds and contact topology in Chapters 1 and 3 respectively, we prove our first result: a link version of the Thurston-Bennequin inequality. This inequality, in its usual formulation, gives an upper bound for the maximal Thurston-Bennequin and self-linking numbers of a knot K in a tight contact 3-manifold (M,ξ), in terms of the Euler characteristic of a Seifert surface for K. A proof of the Thurston-Bennequin inequality for null-homologous knots was given by Eliashberg in [23]. We apply the same strategy, already used by Baker and Etnyre [2], to generalize the inequality to every link L, see Section 3.6, where the resulting upper bound involves the Thurston norm of L.

We define the Thurston norm $\|L\|_T$ for every link in a rational homology sphere in Chapter 2; it is a rational number extracted from the semi-norm, introduced by Thurston in [86], on the relative second homology group of a 3-manifold with toric boundary. A result of Ni [62] and Ozsváth and Szabó [73] implies that link Floer homology detects $\|L\|_T$. The results on the Thurston-Bennequin inequality and the Thurston norm appear in [12].

The computation of link Floer homology in the J-holomorphic setting happens to be very hard. To solve this problem, in [58] Manolescu, Ozsváth, Szabó and Thurston find a combinatorial reformulation of the homology, in terms of grid diagrams. These are grids of squares, that were already known in the 19th century, which can also be used to present links in S^3.

We say that two n-component links in S^3 are strongly concordant if there is a cobordism between them consisting of n disjoint annuli, each one realizing a knot concordance. Then, starting from a grid diagram D of a link L, we define a filtered chain complex $\left(\widehat{GC}(D),\widehat{\partial}\right)$ and we prove in Section 8.3 that its homology, denoted with $\widehat{\mathcal{HFL}}(L)$, is a strong concordance invariant. Our claim is similar to a result of Pardon [74] about Lee homology.

Since we can prove that $\widehat{\mathcal{HFL}}(L)$ has dimension one in Maslov grading zero, we also extract a numerical invariant from the homology group that we call $\tau(L)$ because, in the case of knots, it coincides with the concordance invariant introduced by Ozsváth and Szabó in [69]. The τ-invariant gives a lower bound for the slice genus $g_4(L)$, which is the minimum genus of an oriented, compact surface properly embedded in D^4 and whose boundary is L; see Subsection 8.3.3.

In Chapter 8 we also show that there is a strict relation between the filtration levels of $\widehat{\mathcal{HFL}}(L)$ and the Alexander grading of the torsion-free quotient of $cHFL^-(L)$, a different bigraded version of link Floer homology introduced in [67, 72]. The results about the τ-invariant and link Floer homology appear in [9].

In Chapter 4 we talk in detail about Heegaard diagrams. More specifically, we describe how to construct such a diagram starting from a link L in a 3-manifold M and we show how to relate two diagrams representing smoothly isotopic links in M. As we remarked before, Heegaard diagrams play a crucial role in the definition of Heegaard and link Floer homology, see also Chapter 5, but in the case that M is equipped with a contact structure ξ and L is Legendrian, we need to use a different presentation, which takes into account the contact information.

For this reason we introduce open book decompositions. Given a contact 3-manifold (M,ξ), this is a pair (B,π) where B is a smooth link in M and π is a locally trivial fibration of $M\setminus B$ onto S^1, such that the closures of the fibers have B as boundary. When some compatibility conditions, described in Section 4.2, are satisfied, we say that (B,π) supports the structure ξ.

In [57] Lisca, Ozsváth, Stipsicz and Szabó define open book decompositions (B,π,A), adapted to a Legendrian knot K, by also requiring that K is contained in $S_1=\overline{\pi^{-1}(1)}$ and A is an appropriate basis of the relative first homology group of S_1. In this way they construct the Legendrian knot invariant $\mathfrak{L}(K,M,\xi)$ by applying a result in [47].

We extend this invariant to every Legendrian link L in Section 6.1. This is done by describing a suitable condition for an open book decomposition (B,π,A) to be adapted to L; this time the set A is not a basis anymore, but only a system of generators, the details of the construction are given in Subsection 4.2.1. At this point, we define a special Heegaard diagram D, that we call a Legendrian Heegaard diagram, for the link L in the 3-manifold $-M$, given by reversing the orientation on M; we have that D is obtained up to isotopy from (B,π,A) and then, when L is zero in homology, the invariant $\mathfrak{L}(L,M,\xi)$ is the isomorphism class of a distinguished cycle $\mathfrak{L}(D)$ in the link Floer complex $cCFL^-(D,\mathfrak{t}_\xi)$. In particular, the invariant $\mathfrak{L}(L,M,\xi)$ can be seen as an isomorphism class in the homology group $cHFL^-(-M,L,\mathfrak{t}_\xi)$. All the details can be found in Chapter 6.

We have that $\mathfrak{L}(L,M,\xi)$ is a Legendrian link invariant in the following sense. If L_1 and L_2 are Legendrian isotopic links in (M,ξ) then, given Legendrian Heegaard diagrams D_i representing L_i for $i=1,2$, we can find a chain map between $cCFL^-(D_1)$ and $cCFL^-(D_2)$ which sends $\mathfrak{L}(D_1)$ into $\mathfrak{L}(D_2)$, see Section 6.3.

In [3] Baldwin, Vela-Vick and Vértesi, using a different construction, introduce another invariant of Legendrian links in contact 3-manifolds which generalizes $\mathfrak{L}$ in the case of knots in the standard 3-sphere. The same argument in [3] implies that this invariant coincides with our $\mathfrak{L}$ for every Legendrian link in (S^3,ξ_{st}).

Furthermore, our invariant has the same property of its knot version in [57] regarding loose and non-loose links. We say that a Legendrian link L in an overtwisted contact structure is non-loose if its complement is tight, while it is loose if its complement is overtwisted. We prove in Section 7.1 that $\mathfrak{L}(L,M,\xi)$ vanishes when L is loose. This result gives a sufficient condition for L to be non-loose. In fact, in Chapter 7 it allows us to prove the existence of a non-split, non-loose n-component Legendrian link in every overtwisted structure on S^3. Moreover, using a naturality property of connected sums, we are also able to distinguish two non-loose Legendrian links L_1 and L_2 with the same classical invariants (link type and Thurston-Bennequin and rotation numbers) and Legendrian isotopic components. In other words, we say that the link type of L_1 and L_2 is non-loose non-simple. The results about the invariant $\mathfrak{L}$ and adapted open book decompositions appear in [10].

The invariant $\mathfrak{L}$ seems to give less information in the case of loose links. In fact, we have some results that go in the opposite direction. One of these is a theorem of Dymara, see

[17], which says that two loose knots L_1 and L_2 in (M, ξ), with the same classical invariants, such that there is an overtwisted disk in $M \setminus (L_1 \sqcup L_2)$ are always Legendrian isotopic. In Section 7.2 we generalize this theorem as follows. We relax the condition of having an overtwisted disk disjoint from both knots, but we claim that, in order to have a Legendrian isotopy between L_1 and L_2, we only need the existence of two disjoint overtwisted disks E_1 and E_2 such that E_i is in the complement of L_i for $i = 1, 2$. An example where we can apply our result, but not Dymara's theorem, is the case of a disjoint union of two non-loose knots; in other words, when M is a contact connected sum of M_1 and M_2 and L_i is a non-loose knot in M_i for $i = 1, 2$. The proofs of these results appear in [11].

Finally, in Chapter 9 we give some results on quasi-positive links in S^3. These are links which are obtained as closures of some particular braids, called quasi-positive braids. In particular, we show that we can always compute the maximal self-linking number and the invariant τ of a quasi-positive link.

Moreover, in Section 9.1 we introduce the subfamily of connected transverse $\mathbb{C}$-links: these are links such that the surface Σ_B, associated to a quasi-positive braid B for L, is connected; see Section 9.1. We show that, for this kind of links, the slice genus g_4 is determined by τ. This allows us to prove that the slice genus is additive under connected sums of connected transverse $\mathbb{C}$-links; a result which follows from [44] for knots, but appears to be new in the case of links.

Budapest, March 14 2018

Acknowledgements

I would like to thank Paolo Lisca and András Stipsicz for their advice, the many helpful conversations and the lots of time spent meeting and discussing mathematics.

I want to thank John Etnyre and Brendan Owens for their responsiveness and the help given in these years.

I am grateful to Irena Matkovič for all the time spent together.

After a long time, and maybe a lifetime again, I say thank you to my family.

Budapest, March 21 2018

Chapter 1

Preliminaries

1.1 Knots and links in 3-manifolds

1.1.1 Definition of links

We start with the definition of knots and links. Let us consider a connected, compact, oriented 3-manifold M. Then an *n-component link* is the image of a smooth embedding of a collection of n disjoint circles into M. We usually denote a link with $L \hookrightarrow M$. If we orient all the circles then the link L also inherits an orientation. In this thesis we always assume that a link is oriented, even when this is not explicitly stated. We call a *knot* a link with one component.

Let L_1 and L_2 be two links in a 3-manifold M. Then we say that L_1 is *smoothly isotopic* to L_2 if there exists a smooth map $H : M \times I \to M$ such that

1. $H(\cdot, t)$ is a diffeomorphism for every $t \in I$,
2. $H(\cdot, 0) = Id_M$,
3. $H(L_1, 1) = L_2$.

When two links are smoothly isotopic we also say that they are equivalent. The trivial knot, which is called the *unknot* and is denoted with the symbol $\bigcirc$, is a knot in M that bounds an embedded disk. The unknot is well-defined in every 3-manifold M, in the sense that all such knots are equivalent. See [79].

If a link L is null-homologous in M, which means that the homology class $[L]$ in $H_1(M;\mathbb{Z})$ is zero, then it bounds a compact, oriented embedded surface F inside M. It is easy to see that such a surface can be made connected by adding some tubes and in this case we call F a *Seifert surface* for L. We denote with $g_3(L)$ the minimum genus among all the Seifert surfaces for the link L. Then the following corollary is an immediate consequence of this definition.

Corollary 1.1.1 *The unknot is the only knot K, up to isotopy, such that $g_3(K) = 0$.*

We say that a link $L \hookrightarrow M$ is *split* if L is the disjoint union of L_1 with L_2 and there is a separating, embedded 2-sphere S in M such that S gives the connected sum decomposition $M = M_1 \#_S M_2$ and $L_i \hookrightarrow M_i$ for $i = 1, 2$. When this happens we write $L = L_1 \sqcup L_2$. Otherwise L is called *non-split*. Every link L can be written as $L_1 \sqcup ... \sqcup L_k$ where each L_i is a non-split link in M. The L_i's are called *split components* of L.

We recall that a k-manifold Y is a *rational homology sphere* if $H_i(Y;\mathbb{Q}) \cong H_i(S^k;\mathbb{Z})$ for every $i \in \mathbb{Z}$. Let us take two knots K_1 and K_2 in M and suppose that the 3-manifold M is a rational homology 3-sphere. Then we can define the rational *linking number* between K_1 and K_2 as the rational number

$$\mathrm{lk}_{\mathbb{Q}}(K_1, K_2) = \frac{F \cdot K_2}{t} = \frac{|F \pitchfork K_2|}{t} \ ;$$

where F is a compact, oriented surface, properly embedded in $M_{K_1} = M \setminus \nu(\mathring{K}_1)$ and such that $F \cdot \mu = \mathrm{PD}[\mu][F] = t$, while t is the order of $[K_1]$ in $H_1(M;\mathbb{Z})$. Moreover, here $[\mu]$ and $[K_2]$ are elements of $H_1(M_{K_1};\mathbb{Q}) \cong \mathbb{Q}$. The curve μ represents the meridian of the knot K_1

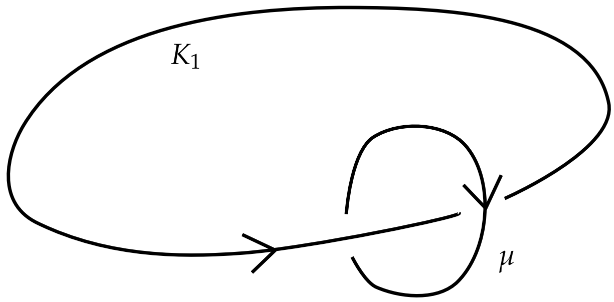


Figure 1.1: The meridian μ is oriented accordingly to the knot K_1.

as shown in Figure 1.1.

Proposition 1.1.2 *The linking number is symmetric. Moreover, if K_1 is null-homologous in M then* $\mathrm{lk}_{\mathbb{Q}}(K_1, K_2)$ *is an integer.*

Proof. The symmetry follows from Poincaré-Lefschetz duality. See [43] for details. On the other hand, the second claim is implied by the fact that, if $t = 1$, the linking number coincides with the algebraic intersection of a Seifert surface for K_1 with the knot K_2; which is clearly an integer. □

In the case when $J = J_1 \cup ... \cup J_n$ and $L = L_1 \cup ... \cup L_m$ are two links in a rational homology sphere M, we define the linking number as

$$\mathrm{lk}_{\mathbb{Q}}(J, L) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{lk}_{\mathbb{Q}}(J_i, L_j) \ . \tag{1.1}$$

We observe that the linking number between two components of a link is invariant under smooth isotopies. Moreover, if J and L are both null-homologous then we omit the subscript $\mathbb{Q}$; thus, the linking number is denoted with $\mathrm{lk}(J, L)$ and it is an integer from Proposition 1.1.2.

1.1.2 Connected sum and mirror image of links

We define an *elementary tangle* as a pair (D^3, α) such that $\alpha \cong I = [0, 1]$ and $\partial D^3 \cap \alpha = \partial\alpha$. See Figure 1.2. An elementary tangle is called trivial if it is isotopic, rel boundary, to $(D^2 \times I, \{0\} \times I)$.

Given two links $L_i \hookrightarrow M_i$ for $i = 1, 2$ we define a *connected sum* $L_1 \# L_2$ as follows. Let D_1, D_2 be two balls such that $(D_i, D_i \cap L_i)$ are trivial elementary tangles. Then we glue the

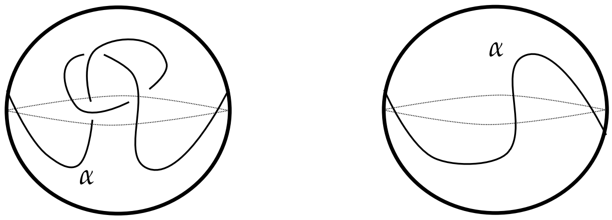


Figure 1.2: Examples of elementary tangles. The one on the right is trivial.

complements of these elementary tangles in M_i using an orientation reversing diffeomorphism $\Phi : \partial D_1 \to \partial D_2$ in the way that $D_1 \cap L_1$ is identified with $D_2 \cap L_2$, respecting the orientations of the links. The result of this operation is a pair $(M_1 \# M_2, L_1 \# L_2)$, where $L_1 \# L_2$ is a connected sum of L_1 and L_2.

The link $L_1 \# L_2$ is well-defined up to the choice of the components of L_i that intersect the trivial elementary tangles in the definition. This means that, if we fix a component K_i in L_i for $i = 1, 2$, the link $L_1 \#_{(K_1,K_2)} L_2$ is determined up to smooth isotopy. Moreover, if L_1 and L_2 are oriented then $L_1 \# L_2$ is also oriented and, from [55], we have the following properties.

Proposition 1.1.3 *Once we fix the components of L_1 and L_2 where we sum the links, the connected sum is commutative and associative.*

Starting from a link $L \hookrightarrow M$, we can always reverse the orientation of the 3-manifold M and then L defines a new link when seen as embedded in $-M$. Now, suppose that the manifold M admits a diffeomorphism $i_- : M \to M$ which reverses the orientation. Then in this case the oriented link $L^* = i_-(L)$ is called the *mirror image* of L.

1.2 Links in the 3-sphere

1.2.1 Diagrams for links in S^3

Every link embedded in S^3 can be represented by a planar diagram. In fact, first we

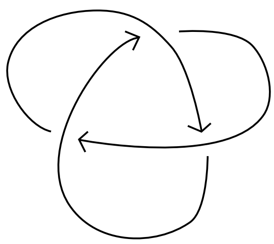


Figure 1.3: An oriented diagram for the positive trefoil knot.

observe that a link $L \hookrightarrow S^3$ is also embedded in $\mathbb{R}^3$; then we choose an affine 2-subspace $H \subset \mathbb{R}^3$, which is diffeomorphic to $\mathbb{R}^2$, such that for the corresponding orthogonal projection $p_H : \mathbb{R}^3 \to H$ is

- $|p_H^{-1}(x) \cap L| \leqslant 1$ for every $x \in H$ except a finite number of points, called *singular values*, where this number is equal to 2;
- the lines tangent to L in the singular values never project to the same line in H.

A proof of the fact that the plane H exists for every link L in S^3 can be found in [55]. At this point we define a diagram D for L as the image of L under the projection p_H. Moreover,

Figure 1.4: The signs of a crossing in a diagram.

we say that

- a crossing in D is a singular value;
- an arc in D is a line which connects two crossings, possibly the same, one as starting point and the other one as end point;
- each crossing contains the information of which arcs are overcrossings or undercrossings.

When the link is oriented, a sign can be assigned to each crossing in the diagram D, as shown in Figure 1.4.

A theorem of Reidemeister assures us that two links are equivalent if and only their diagrams are related by some moves, which are called Reidemeister moves. There are

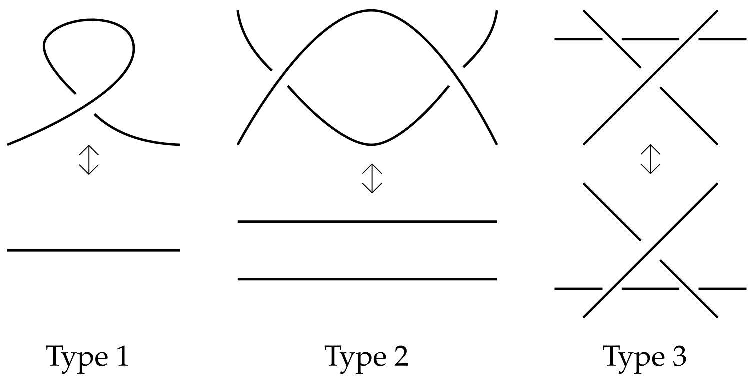


Figure 1.5: The three Reidemister moves. For each move, we have to consider all the possible combinations of reflections and orientations.

three types of these moves and they are pictured in Figure 1.5.

Theorem 1.2.1 (Reidemeister) *Two links L_1 and L_2 in S^3 are smoothly isotopic if and only if, given D_1, D_2 diagrams for L_1, L_2 respectively, the two diagrams D_1 and D_2 differ by a finite sequence of Reidemeister moves and planar isotopies.*

A proof of this theorem is found in [55].

1.2.2 The Seifert algorithm

In the 3-sphere we have a method which allows us to determine a Seifert surface for a given oriented link L, starting from a diagram D. We define the *oriented resolution* of D as the collection of circles in $\mathbb{R}^2$ obtained by resolving all the crossings in D preserving the orientation. It is easy to see that, for each crossing, this can be done only in one possible way, as shown in Figure 1.6. The circles have the orientation induced on them by D.

From the oriented resolution we can construct a compact, oriented surface F in the 3-space which is bounded by the link L. We start by pushing all the circles up, starting from the innermost ones, until they are all on different levels and then we take the disks that they bound. So we now have a collection of disjoint disks in $\mathbb{R}^3$. We connect these

Figure 1.6: Examples of orientation preserving resolutions.

disks by attaching negative (positive) bands in correspondence with the positive (negative) crossings in D. The orientation on F is defined in the following way: it coincides with the one induced by $\mathbb{R}^2$ on the circles oriented counter-clockwise, while it is the opposite on the circles oriented clockwise. It is easy to check that such orientation can be extended to the whole surface F.

Since we can suppose that the resolutions are always performed in a very small neighborhood of the crossing, we can connect two circles, in the oriented resolution of D, by adding lines in correspondence with each crossings. In this way we define a planar subspace $\mathcal{D}$ associated to the oriented resolution, as in Figure 1.7. The subspace $\mathcal{D}$ is homotopy equivalent to the 4-valent graph given by D when we remove the under and over-crossings. Moreover, the surface F is connected if and only if its associated planar subspace is connected. The latter property, in knot theory, is the definition of *non-split diagram* of a

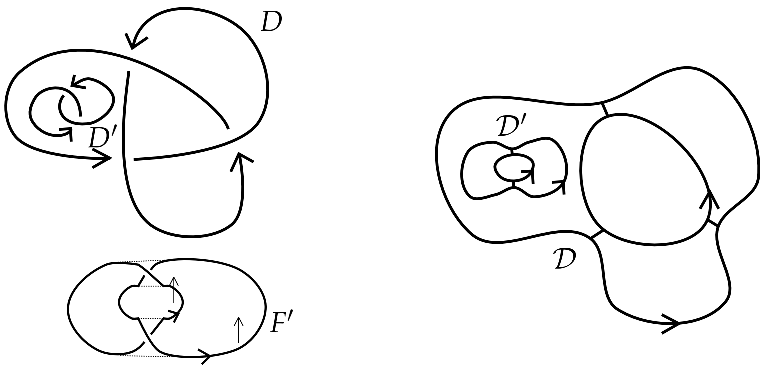


Figure 1.7: A diagram and its associated oriented resolution. The surface F' is obtained by applying the given algorithm to D'.

link; moreover, it implies the following theorem.

Theorem 1.2.2 (Seifert algorithm) *Every diagram D of a link in S^3 gives a Seifert surface by applying the procedure described above.*

Proof. When D is non-split, the surface F determined by the oriented resolution of the diagram is a Seifert surface for L for what we say before. If D is split then we need to connect the components of F together by removing small disks and inserting long, thin tubes; at this point the claim follows easily. □

1.2.3 Operations on links and planar diagrams

In Subsection 1.1.1 we defined the linking number between two knots. In S^3 such

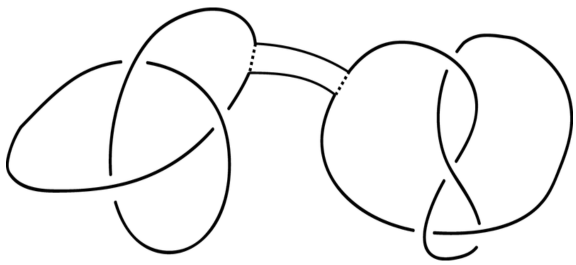


Figure 1.8: A diagram for the connected sum between a negative trefoil knot and a figure-eight knot.

number is always an integer, since the group $H_1(S^3;\mathbb{Z})$ is trivial, and it is computed easily using knot diagrams .

Proposition 1.2.3 *Let D_1, D_2 be diagrams for the oriented knots K_1, K_2 respectively. Then*

$$\mathrm{lk}(K_1, K_2) = \sum_{p \mid D_1 \uparrow D_2} \epsilon_p \,,$$

where the symbol $\uparrow$ means that, at the crossing p, the arc in D_1 overcrosses the one in D_2 and ϵ is the sign of p, according to Figure 1.4.

Proof. It follows easily from the definition of linking number; see [79]. □

Given a diagram D, we can also define the *writhe* of D, and we denote it with $\mathrm{wr}(D)$, as

$$\mathrm{wr}(D) = \sum_p \epsilon_p \,,$$

where ϵ and p are as in Proposition 1.2.3. Here the sum is taken over all the crossings in D. The writhe is not a link invariant, but it has the following property.

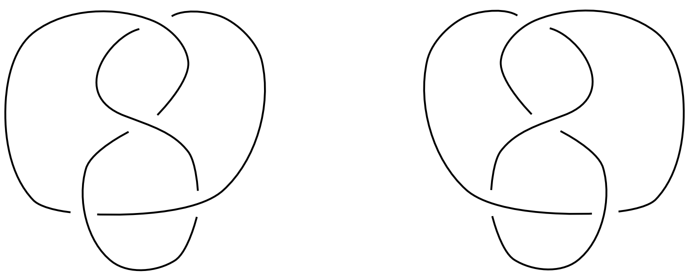


Figure 1.9: A diagram for the Figure-eight knot and its mirror image. It is easy to see that the two knots are isotopic.

Proposition 1.2.4 *The writhe of the diagram D is invariant under Reidemeister 2 and 3 moves, while it changes by ± 1 under Reidemeister 1 moves.*

Proof. It is enough to check that the claim holds for the moves in Figure 1.5. □

Figure 1.8 shows how the connected sum between two links can be seen from planar diagrams . On the other hand, a diagram for the mirror image of L is gotten from a diagram of L by changing the sign of all the crossings. See Figure 1.9.

1.3 Cobordisms

1.3.1 Definition of cobordism and the slice genus

We start this section by giving the definition of cobordism for links in S^3. A genus g

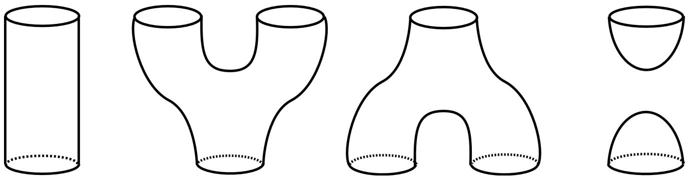


Figure 1.10: The standard cobordisms in $S^3 \times I$. Each one contains at most one critical point.

cobordism, between links L_1 and L_2 in the 3-sphere, is the image of a smooth embedding $f : \Sigma_g \to S^3 \times I$, where Σ_g is a compact, oriented surface of genus g; more precisely, the surface Σ_g has connected components $\Sigma_{g_1}, ..., \Sigma_{g_J}$ and $g = g_1 + ... + g_J$. Furthermore, we

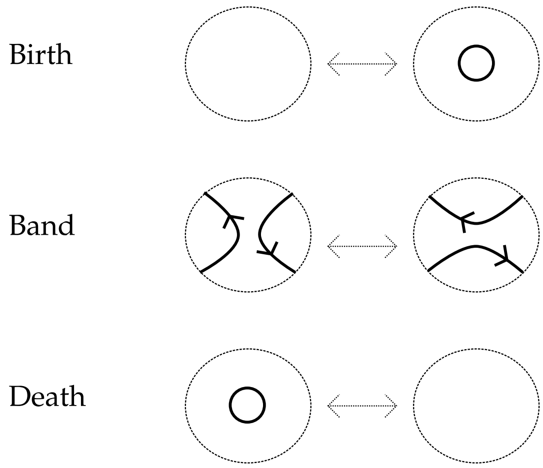


Figure 1.11: The three Morse moves. Birth and Death moves consist of creating and deleting disjoint unknots.

have that Σ_g satisfies the following properties:

1. $f(\partial \Sigma_g) = (-L_1) \times \{0\} \sqcup L_2 \times \{1\}$;
2. $f(\Sigma_g \setminus \partial \Sigma_g) \subset S^3 \times (0, 1)$;
3. every connected component of Σ_g has boundary in both L_1 and L_2.

In all the figures in this section cobordisms are drawn as standard surfaces in S^3, but they can be knotted in $S^3 \times I$.

It is a standard result in Morse theory, see [59], that a link cobordism can be decomposed into five standard cobordisms, which appear in Figure 1.10. Then this gives a generalization of Reidemeister theorem.

Theorem 1.3.1 *Let D_1 and D_2 be two diagrams for the links L_1 and L_2 in S^3. Then for every cobordism Σ, between L_1 and L_2, the diagram D_2 is obtained from D_1 by a finite number of Reidemeister and Morse moves, where the latter ones are pictured in Figure 1.11.*

In particular, there is a cobordism between L_1 and L_2 in S^3 that does not have any critical point if and only if L_1 is smoothly isotopic to L_2.

We say that a cobordism is a *strong cobordism* if the links have the same number of components and, together with the properties stated above, the surface Σ_g is such that each connected component Σ_{g_i} determines a knot cobordism. Figure 1.12 shows an example of

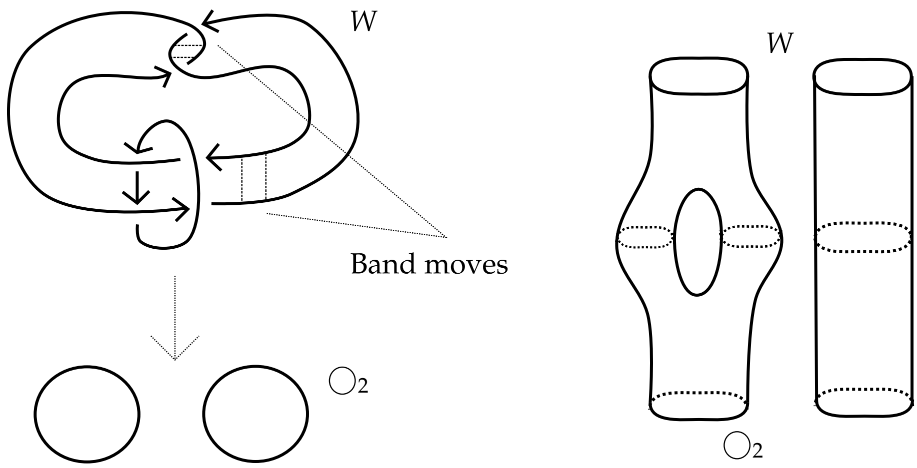


Figure 1.12: The Whitehead link W can be changed into the 2-component unlink by performing two Band moves. Note that one component of W remains untouched.

this type of cobordism.

If there is a genus zero cobordism between a link J and the unknot then J is called *weakly slice*. This is equivalent to say that J bounds a properly embedded punctured disk in D^4. Since $H_1(D^4, S^3; \mathbb{Z})$ is trivial, we conclude that there is always a compact, oriented surface, which is properly embedded in D^4 and bounds a given link L. We define the *slice genus* of L as the minimum genus of a surface of this kind and we denote it with $g_4(L)$. Obviously, it follows that $g_4(L) \leqslant g_3(L)$.

1.3.2 Link concordance

Two n-component links L_1 and L_2 are *concordant*, or *strongly concordant*, if there exists a genus zero strong cobordism between them. In other words, when the cobordism Σ consists of n disjoint annuli and the boundary of each of these is the union of a component of L_1 and one of L_2. Theorem 1.3.1 implies the following corollary.

Corollary 1.3.2 *Suppose that L_1 is smoothly isotopic to L_2 as a link in S^3. Then L_1 and L_2 are concordant.*

If a link L is concordant to the n-component unlink then we say that L is *slice*, or *strongly slice*. In particular, the link L is slice if and only if it bounds n disjoint disks, properly embedded in D^4.

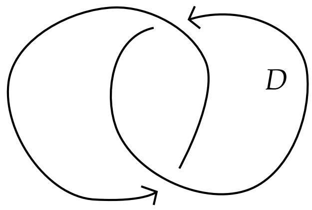


Figure 1.13: A diagram for the positive Hopf link.

In the case of knots, we immediately observe that the notions of weakly and strongly slice coincide. This is no longer true when we consider links with at least two components. In fact, let us take the positive Hopf link H_+, which is represented by a the diagram D in Figure 1.13. Then H_+ is weakly slice, because the surface obtained by applying the Seifert algorithm to D is diffeomorphic to an annulus. On the other hand, it cannot be strongly slice since, say K_1 and K_2 are its components, we have that $\text{lk}(K_1, K_2) = 1$ and this would contradict the following proposition.

Proposition 1.3.3 *If a link $L = L_1 \cup ... \cup L_n$ is strongly cobordant to an unlink then $\text{lk}(L_i, L_j) = 0$ for every $i \neq j$. As a consequence, the linking numbers between the components of a link are concordance link invariants.*

Proof. The claim can be proved by using Proposition 1.2.3 and Theorem 1.3.1. □

We saw in Subsection 1.1.2 that the connected sum is a well-defined, associative and commutative operation on the set of knots, up to isotopy. However, this operation does not give rise to a group because a non-trivial knot cannot be inverted. This changes when we consider the relation of knot concordance. In fact, we now have the following theorem.

Theorem 1.3.4 *The set of all the knots in S^3 up to concordance, equipped with the operation given by the connected sum, is an abelian group that it is called smooth knot concordance group and it is denoted with $\mathcal{C}_1$.*

Proof. We have that the connected sum is well-defined up to concordance; this follows from the properties of the connected sum and the fact that if K and L are two concordant knots then $K \# -L^*$ is slice, which is proved in [55], where $-L^*$ denotes the mirror image of the knot obtained from L by reversing the orientation.

Therefore, if $[K]$ is the concordance class of K, we can define

$$[K] + [L] = [K \# L], \quad -[K] = [-K^*] \quad \text{and} \quad 0 = [\bigcirc]$$

for every knot K and L and where $\bigcirc$ is the unknot. It is now easy to check that $\mathcal{C}_1$ is indeed an abelian group. □

The algebraic structure of the group $\mathcal{C}_1$ has not been determined completely, but there are some results in this direction. In [34] Fox and Milnor prove that $\mathcal{C}_1$ has 2-torsion, by showing that the Figure-eight knot, see Figure 1.9, which is isotopic, and then concordant, to its reverse mirror image is not a slice knot. Moreover, the work of Levine in [53, 54] tells us that the knot concordance group has a direct summand isomorphic to $\mathbb{Z}^\infty$ and then it is not finitely generated.

1.4 Filtered chain complexes

1.4.1 Filtered spaces ...

For the purposes of this background section, we fix a field $\mathbb{K}$. Let us consider a graded chain complex $\mathcal{C} = (C, \partial)$, where C is a $\mathbb{K}$-vector space and $\partial : C \to C$ is a linear map, satisfying the following properties:

- there is a splitting of C as a $\mathbb{K}$-vector space $C = \bigoplus_{d \in \mathbb{Z}} C_d$. In other words, the space C is graded;
- the differential ∂ is such that $\partial \circ \partial = 0$ and it is compatible with the grading, in the sense that $\partial(C_d) \subset C_{d-1}$.

We equip $\mathcal{C}$ with a sequence of $\mathbb{K}$-subspaces $\mathcal{F}^s C \subset C$ with $\mathcal{F}^s C \subset \mathcal{F}^{s+1} C$ for every $s \in \mathbb{Z}$, which is such that

$$\bigcup_{s \in \mathbb{Z}} \mathcal{F}^s C = C\,.$$

This collection of subspaces is called a *filtration* on $\mathcal{C}$ if the following conditions hold:

- if we define $\mathcal{F}^s C_d = (\mathcal{F}^s C) \cap C_d$ then it is $\mathcal{F}^s C = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^s C_d$;
- the filtration is compatible with the differential, which means that $\partial(\mathcal{F}^s C) \subset \mathcal{F}^s C$;
- for every $d \in \mathbb{Z}$ there is an s_d such that $\mathcal{F}^{s_d} C_d = \{0\}$.

A complex $\mathcal{C}$ as above is called *filtered chain complex*. Moreover, the *filtration level* of a non-zero element $x \in C$ is the minimal s for which $x \in \mathcal{F}^s C \subset C$.

We now define the *graded object* associated to a filtered complex $\mathcal{C}$ to be the bigraded chain complex $\big(\mathrm{gr}(C), \mathrm{gr}(\partial)\big)$, where

$$\mathrm{gr}(C)_{d,s} = \frac{\mathcal{F}^s C_d}{\mathcal{F}^{s-1} C_d}$$

and $\mathrm{gr}(\partial)$ is the map induced by ∂ on $\mathrm{gr}(C)$. We denote the associated graded object with $\mathrm{gr}(\mathcal{C})$.

The filtered chain complex $\mathcal{C}$ and its associated graded object, which is a bigraded chain complex, $\mathrm{gr}(\mathcal{C})$ give us two homology groups. In the second case the space is constructed easily. In fact, we say that

$$H_{*,*}(\mathrm{gr}(\mathcal{C})) = \bigoplus_{d,s \in \mathbb{Z}} H_{d,s}(\mathrm{gr}(\mathcal{C})) = \frac{\mathrm{Ker}\,\big(\mathrm{gr}(\partial)_{d,s}\big)}{\mathrm{Im}\,\big(\mathrm{gr}(\partial)_{d+1,s}\big)}$$

and the result is a bigraded space.

On the other hand, the complex $\mathcal{C}$ is graded and then we can define the homology group

$$H_*(\mathcal{C}) = \bigoplus_{d \in \mathbb{Z}} H_d(\mathcal{C}) = \frac{\mathrm{Ker}\,\partial_d}{\mathrm{Im}\,\partial_{d+1}}\,,$$

which is a graded $\mathbb{K}$-vector space. Moreover, since $\mathcal{C}$ is also equipped with a filtration $\mathcal{F}$, we can find a way to induce $\mathcal{F}$ on $H_*(\mathcal{C})$. More specifically, we introduce the subspaces

$\mathcal{F}^s H_d(\mathcal{C})$ as follows: consider the projection $\pi_d : \text{Ker } \partial_d \to H_d(\mathcal{C})$. Denote with Ker $\partial_{d,s}$ the subspace Ker $\partial_d \cap \mathcal{F}^s C_d$; we say that

$$\mathcal{F}^s H_d(\mathcal{C}) = \pi_d(\text{Ker } \partial_{d,s})$$

for every $s \in \mathbb{Z}$. Thus, the fact that Ker $\partial_{d,s} \subset$ Ker $\partial_{d,s+1}$ implies that the filtration $\mathcal{F}$ descends to homology. We can extend the filtration $\mathcal{F}$ on the total homology $H_*(\mathcal{C})$ by taking

$$\mathcal{F}^s H_*(\mathcal{C}) = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^s H_d(\mathcal{C}) \,.$$

1.4.2 ... and filtered maps

Fix two filtered, graded chain complexes $\mathcal{C} = (C, \partial)$ and $\mathcal{C}' = (C', \partial')$ over $\mathbb{K}$. A chain map $f : C \to C'$ is a linear map such that $\partial' \circ f = f \circ \partial$. A chain map *preserves the grading* if $f(C_d) \subset C'_d$ and it is *filtered of degree t* if $f(\mathcal{F}^s C) \subset \mathcal{F}^{s+t} C'$ for every $d, s \in \mathbb{Z}$. A filtered chain map induces a map in homology that is filtered of the same degree. This means that f induces a map $f_* : H_*(\mathcal{C}) \to H_*(\mathcal{C}')$ such that $f_*(\mathcal{F}^s H_*(C)) \subset \mathcal{F}^{s+t} H_*(\mathcal{C}')$ for every $s \in \mathbb{Z}$.

Now, given two chain maps $f, g : C \to C'$ filtered of degree t, such that they preserve the gradings, a *filtered chain homotopy* from g to f is a map $H : C \to C'$ that satisfies the following properties:

- H maps $\mathcal{F}^s C_d$ into $\mathcal{F}^{s+t} C'_{d+1}$;
- H satisfies the homotopy relation

$$\partial' \circ H + H \circ \partial = f - g \,; \tag{1.2}$$

- H is a linear map.

Two maps as the ones above are said to be filtered chain homotopic.

Two chain complexes $\mathcal{C}$ and $\mathcal{C}'$ are *filtered chain homotopy equivalent* if there are maps $f : C \to C'$ and $g : C' \to C$, which preserve the gradings and are filtered of degree zero, with the property that the maps $f \circ g$ and $g \circ f$ are filtered chain homotopic to the respective identity maps. In this case, the map f is called a *filtered chain homotopy equivalence*; moreover, the maps f and g are said to be filtered chain homotopy inverses of one another.

A filtered chain map f naturally induces a chain map $\text{gr}(f) : \text{gr}(C) \to \text{gr}(C')$ between the associated graded objects. It is easy to see that, if $f : C \to C'$ is a filtered chain homotopy equivalence, the map $\text{gr}(f)$ is also a chain homotopy equivalence.

Furthermore, we call *filtered quasi-isomorphism* a chain map $f : C \to C'$, which is again filtered of degree zero and preserves the grading, whose associated graded map $\text{gr}(f)$ induces an isomorphism between the homology groups $H_{*,*}(\text{gr}(\mathcal{C}))$ and $H_{*,*}(\text{gr}(\mathcal{C}'))$. We say that $\mathcal{C}$ and $\mathcal{C}'$ are *filtered quasi-isomorphic* if there exists a third complex $\mathcal{C}''$ and filtered quasi-isomorphisms from $\mathcal{C}''$ to $\mathcal{C}$ and from $\mathcal{C}''$ to $\mathcal{C}'$. Then, from [67], the following proposition holds.

Proposition 1.4.1 *Two filtered, graded chain complexes $\mathcal{C}$ and $\mathcal{C}'$ over a field $\mathbb{K}$ are filtered quasi-isomorphic if and only if they are filtered chain homotopy equivalent.*

Because of this result, we always write that two filtered chain complexes are filtered chain homotopy equivalent, instead of saying that they are quasi-isomorphic.

We say that a linear map $F : H_*(\mathcal{C}) \to H_*(\mathcal{C}')$ is a *filtered isomorphism* if F and its inverse are both filtered of degree zero. In general, we also require that such isomorphism preserves the grading; we explicitly write when this does not happen. The following corollary follows immediately from this definition.

Corollary 1.4.2 *The map F as above is a filtered isomorphism if and only if*

$$\mathcal{F}^s H_d(\mathcal{C}) \cong_{\mathbb{K}} \mathcal{F}^s H_d(\mathcal{C}')$$

for every $d, s \in \mathbb{Z}$.

Furthermore, we can prove the following proposition.

Proposition 1.4.3 *If* $\mathcal{C}$ *and* $\mathcal{C}'$ *are filtered chain homotopy equivalent complexes over* $\mathbb{K}$ *then* $H_{*,*}(gr(\mathcal{C})) \cong H_{*,*}(gr(\mathcal{C}'))$*, as bigraded* $\mathbb{K}$*-vector spaces, and* $H_*(\mathcal{C})$ *is filtered isomorphic to* $H_*(\mathcal{C}')$.

Proof. The first implication follows from Proposition 1.4.1 and the definition of quasi-isomorphism. For the second implication, say ϕ is a filtered chain homotopy equivalence between $\mathcal{C}$ and $\mathcal{C}'$, we notice that ϕ has a filtered chain homotopy inverse, that we call ψ, and, since the homotopy condition in Equation (1.2) holds, we have that the induced maps in homology ϕ_* and ψ_* are inverses one of the other. Now, we just need to observe that ϕ and ψ are both filtered of degree zero and then this holds for ϕ_* and ψ_* too. This implies precisely that ϕ_* is a filtered isomorphism. □

Let $f : C \to C'$ be a chain map, which is filtered of degree t and preserves the grading. We can define its *associated mapping cone* , that is denoted with $\mathrm{Cone}(f : C \to C')$, whose underlying filtered chain complex is exactly

$$\mathcal{F}^s(C \oplus C')_d = \mathcal{F}^{s-t}C_{d-1} \oplus \mathcal{F}^s C'_d \,;$$

while the differential is given by

$$\partial_{\mathrm{Cone}} : (C \oplus C')_d \to (C \oplus C')_{d-1}$$
$$\partial_{\mathrm{Cone}}(x, y) = (-\partial(x), \partial(y) + f(x)) \,.$$

Then we have the following result about maps between mapping cones, whose proof appears in [67].

Lemma 1.4.4 *Let* $\mathcal{C}, \mathcal{C}', \mathcal{E}, \mathcal{E}'$ *be four filtered, graded chain complexes over* $\mathbb{K}$*. Suppose that there are filtered chain maps* f *and* g *as in the following square.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\phi"'] & \mathcal{C}' \arrow[d, "\phi'"] \\ \mathcal{E} \arrow[r, "g"] & \mathcal{E}' \end{tikzcd}$$

Moreover, we can find two filtered chain homotopy equivalences ϕ *and* ϕ' *such that* $\phi' \circ f$ *is filtered chain homotopic to* $g \circ \phi$*. Then* $\mathrm{Cone}(f)$ *and* $\mathrm{Cone}(g)$ *are filtered homotopy equivalent.*

This lemma will be useful in the remaining of the thesis. In particular, we use it to prove a result in Chapter 8.

1.5 Tensor product of modules

1.5.1 General properties and vector spaces

In this section we recall the definition of tensor product of R-modules, where R is a commutative ring. More details can be found in [1].

Let M and N be two R-modules over the commutative ring R. Then the *tensor product* of M with N over R is an R-module $M \otimes_R N$, together with the canonical R-bilinear map

$$\otimes : M \oplus N \longrightarrow M \otimes_R N\,,$$

which is universal in the following sense: for every R-module L and every R-bilinear map $f : M \oplus N \to L$, there is a unique R-linear map

$$\phi : M \otimes_R N \longrightarrow L$$

such that $\phi \circ \otimes = f$; which means that the following diagram commutes.

$$\begin{tikzcd} M \oplus N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

The fact that the R-module $M \otimes_R N$ exists and it is unique, up to a canonical isomorphism, is proved in [1]. Moreover, we have the following canonical isomorphisms:

1. identity:
$$R \otimes_R M = M\,; \tag{1.3}$$

2. associativity:
$$(M \otimes_R N) \otimes_R P = M \otimes_R (N \otimes_R P)\,; \tag{1.4}$$

3. symmetry:
$$M \otimes_R N = N \otimes_R M\,; \tag{1.5}$$

4. distributivity:
$$M \otimes_R (N \oplus P) = (M \otimes_R N) \oplus (M \otimes_R P)\,. \tag{1.6}$$

The following proposition can also be found in [1].

Proposition 1.5.1 (Right exactness) *If the following sequence of R-modules*

$$0 \longrightarrow N_1 \xrightarrow{f} N \xrightarrow{g} N_2 \longrightarrow 0$$

is exact, then

$$M \otimes_R N_1 \xrightarrow{1\otimes f} M \otimes_R N \xrightarrow{1\otimes g} M \otimes_R N_2 \longrightarrow 0$$

is also an exact sequence, where $(1 \otimes f)(x \otimes y) = x \otimes f(y)$.

From these properties we can immediately compute the tensor product of finite dimensional vector spaces over the field $\mathbb{K}$.

Proposition 1.5.2 *If V and W are $\mathbb{K}$-vector spaces, whose dimensions are equal to n and m respectively, then we have that $V \otimes W$ is isomorphic to $\mathbb{K}^{nm}$. Furthermore, if $\mathcal{B}_1 = \{v_1, ..., v_n\}$ and $\mathcal{B}_2 = \{w_1, ..., w_m\}$ are basis for V and W then $\{v_1 \otimes w_1, ..., v_n \otimes w_m\}$ is a basis for $V \otimes W$.*

Proof. We have that $V \cong_{\mathbb{K}} \mathbb{K}^n$ and $W \cong_{\mathbb{K}} \mathbb{K}^m$. Then, using Equations (1.4), (1.5) and (1.6), we obtain

$$V \otimes W \cong_{\mathbb{K}} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (\mathbb{K} \otimes \mathbb{K}) \,,$$

which is isomorphic to $\mathbb{K}^{nm}$ by Equation (1.3). $\square$

1.5.2 Finitely generated $\mathbb{K}[x]$-modules

Suppose now that $R = \mathbb{K}[x]$, where $\mathbb{K}$ is a field. Since $\mathbb{K}[x]$ is a principal ideal domain, from [1] we know that it is in particular a Dedekind domain and this implies that a $\mathbb{K}[x]$-module is the sum of cyclic submodules. More specifically, if M is a module over $\mathbb{K}[x]$ and it is finitely generated then we can write

$$M \cong_{\mathbb{K}[x]} \mathbb{K}[x]^r \oplus \left(\bigoplus_{i=1}^{n} \frac{\mathbb{K}[x]}{f_i(x)\mathbb{K}[x]} \right) \,;$$

where each $f_i(x)$ is a non-zero polynomial in $\mathbb{K}[x]$ and $r, n \geqslant 0$. Therefore, we are able to prove the following result .

Proposition 1.5.3 *Let us consider M and N two finitely generated $\mathbb{K}[x]$-modules presented as follows:*

$$M \cong_{\mathbb{K}[x]} \bigoplus_{i=1}^{n} \frac{\mathbb{K}[x]}{f_i(x)\mathbb{K}[x]} \quad \textit{and} \quad N \cong_{\mathbb{K}[x]} \bigoplus_{j=1}^{m} \frac{\mathbb{K}[x]}{g_j(x)\mathbb{K}[x]} \,,$$

where $f_i(x), g_j(x) \in \mathbb{K}[x]$ for every $i = 1, ..., n$ and $j = 1, ..., m$. Then the tensor product of M with N over $\mathbb{K}[x]$ is given by

$$M \otimes_{\mathbb{K}[x]} N \cong_{\mathbb{K}[x]} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \frac{\mathbb{K}[x]}{gcd(f_i(x), g_j(x))\mathbb{K}[x]} \,.$$

Before starting the proof, we need a lemma.

Lemma 1.5.4 *If I and J are two ideals of R then*

$$R/I \otimes_R R/J = R/I + J \,.$$

Proof. Tensoring with an R-module M and using Proposition 1.5.1, the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

gives the exact sequence

$$I \otimes_R M \xrightarrow{f} R \otimes_R M = M \longrightarrow R/I \otimes_R M \longrightarrow 0 \,,$$

where f is given by $i \otimes x \mapsto ix$. From the fact that the image of f is the submodule IM, we then obtain that

$$R/_{I} \otimes_R M = M/_{IM}.$$

Hence, the claim follows once we put R/J in place of M. $\square$

Now we can go back to Proposition 1.5.3.

Proof of Proposition 1.5.3. We apply the properties of the tensor product that we stated in Subsection 1.5.1 and we obtain that, in order to prove the statement, it is enough to show that the following relation

$$\frac{\mathbb{K}[x]}{f(x)\mathbb{K}[x]} \otimes_{\mathbb{K}[x]} \frac{\mathbb{K}[x]}{g(x)\mathbb{K}[x]} \cong_{\mathbb{K}[x]} \frac{\mathbb{K}[x]}{\gcd(f(x), g(x))\mathbb{K}[x]}$$

holds for every $f(x), g(x) \in \mathbb{K}[x]$. This is done easily from Lemma 1.5.4 by taking $R = \mathbb{K}[x]$, $I = (f(x))$ and $J = (g(x))$. $\square$

Chapter 2

The Thurston norm

2.1 Definition

Thurston in [86] introduced a semi-norm on the homology of some 3-manifolds. In this chapter we recall the construction in the specific case of rational homology spheres.

Let us consider a compact, connected, oriented 3-manifold Y such that its boundary consists of some tori. We call the *complexity* of a compact, oriented surface F properly embedded in $(Y, \partial Y)$, the integer

$$\chi_-(F) = -\sum_{i=1}^{m} \chi(F_i)\,,$$

where $F_1, \ldots, F_m$ are the connected components of F which are not closed and not diffeomorphic to disks. If m happens to be zero then we say that the complexity is also zero. Then we define the function

$$\|\cdot\|_Y : H_2(Y, \partial Y; \mathbb{Z}) \longrightarrow \mathbb{Z}_{\geqslant 0}$$

by requiring that

$$\|a\|_Y = \min\{\chi_-(F)\}\ ,$$

where F is a surface as above that represents the relative homology class $a \in H_2(Y, \partial Y; \mathbb{Z})$.

It can be shown [7] that

1. $\|la\|_Y = |l| \cdot \|a\|_Y$ for any $l \in \mathbb{Z}$ and $a \in H_2(Y, \partial Y; \mathbb{Z})$,
2. $\|la + mb\|_Y \leqslant |l| \cdot \|a\|_Y + |m| \cdot \|b\|_Y$ for any $l, m \in \mathbb{Z}$ and $a, b \in H_2(Y, \partial Y; \mathbb{Z})$.

Suppose from now on that M is a rational homology 3-sphere and L is a smooth link in M. This implies that $H_2(M_L, \partial M_L; \mathbb{Q}) \cong \mathbb{Q}^n$, where $M_L = M \setminus \nu(\mathring{L})$ and n is the number of components of L. Then the *Thurston semi-norm*

$$\|\cdot\|_{M_L} : H_2(M_L, \partial M_L; \mathbb{Q}) \longrightarrow \mathbb{Q}_{\geqslant 0}$$

is defined as before for integer homology classes and extended to the whole $\mathbb{Q}^n$ by saying that $\|la\|_{M_L} = |l| \cdot \|a\|_{M_L}$ for every $l \in \mathbb{Q}$ and $a \in H_2(M_L, \partial M_L; \mathbb{Q})$.

If L is a null-homologous n-component link in M then we can easily extract a number from $\|\cdot\|_{M_L}$. Namely, we define the Thurston norm of L as the integer

$$\|L\|_T = \|[F]\|_{M_L}\ ,$$

where F is a Seifert surface for L. See Subsection 1.1.1 for the definition of Seifert surfaces. While $\|\cdot\|_{M_L}$ is effectively a semi-norm, the integer $\|\cdot\|_T$ is just a numerical invariant of the link L.

Lemma 2.1.1 *In a rational homology sphere M, all the Seifert surfaces for a null-homologous n-component link L represent the same relative homology class in M_L.*

Proof. From Poincaré-Lefschetz duality, see [43], we have that $H_2(M_L, \partial M_L; \mathbb{Z}) \cong H^1(M_L; \mathbb{Z})$ and the latter group is isomorphic to $\mathbb{Z}^n$, since M is a rational homology sphere. Moreover, duality gives an identification between the $\mathbb{Z}$-vector $(a_1, ..., a_n)$ and all the compact, oriented, properly embedded surfaces F in M_L such that $F \cdot \mu_i = a_i$ for every $i = 1, ..., n$, where μ_i is the meridian of the i-th component of L, see Figure 1.1. This happens because L is null-homologous in M.

The meaning of our claim is that such surfaces represent the same relative homology class. Therefore, since in the case when F is a Seifert surface we have that $F \cdot \mu_i = 1$ for each i, the proof is completed because Seifert surfaces satisfy by definition all the properties that we required. □

It is clear that this lemma implies that the Thurston norm $\|L\|_T$ is well-defined; in fact, the value of the semi-norm $\|\cdot\|_{M_L}$ depends only on the relative homology class of the surface.

In [73] Ozsváth and Szabó prove that the link Floer homology group $\widehat{HFL}(L)$ of a link in S^3, introduced in [69, 72], detects the Thurston norm. This result has been generalized to null-homologous links in rational homology spheres by Ni [62].

Theorem 2.1.2 *Suppose that L is a null-homologous n-component link in a rational homology 3-sphere M. Then we have that*

$$\max\left\{s \in \mathbb{Q} \mid \widehat{HFL}_{*,s}(L) \neq \{0\}\right\} = \frac{\|L\|_T - o(L) + n}{2},$$

where $o(L)$ denotes the number of disjoint unknots in L.

The definition of the Thurston norm $\|\cdot\|_T$ does not immediately extend to links that are not null-homologous; in fact, these links do not admit Seifert surfaces. In order to avoid this problem we need to introduce rational Seifert surfaces, see also [2].

Let us consider an n-component link L with *order t* in M. This means that $[L]$ has order t in the group $H_1(M; \mathbb{Z})$, which is finite because M is a rational homology sphere. Then F is *rationally bounded* by L if there is a map $j : \Sigma \to M$, where Σ is a compact, oriented surface with no closed components, such that

- $j(\Sigma) = F$;
- $j|_{\mathring{\Sigma}}$ is an embedding of the interior of Σ in $M \setminus L$;
- $j|_{\partial\Sigma}$: $\partial\Sigma \to L$ is a t-fold cover of all the components of L.

Moreover, if F is also connected then we call it a *rational Seifert surface* for L.

Let us consider again the meridian curves $\{\mu_1, ..., \mu_n\}$ of L as in Figure 1.1; where μ_i is embedded in $\partial\nu(L_i)$ and L_i is the i-th component of L. As we remark in the proof of Lemma 2.1.1, Poincaré-Lefschetz duality gives that two properly embedded surfaces F_1 and F_2 in M_L represent the same relative homology class if and only if $F_1 \cdot \mu_i = F_2 \cdot \mu_i$ for every $i = 1, ..., n$, where as before we mean the algebraic intersection of the surface F_j with the curve μ_i in the 3-manifold M_L.

Lemma 2.1.3 *Suppose that F is a compact, oriented surface properly embedded in $M_L = M \setminus \mathring{\nu(L)}$ with $F \cdot \mu_i = t$ for every $i = 1, ..., n$, where M is a rational homology sphere and $L \hookrightarrow M$ is an n-component link with order t.*

Then there exists an F' in M which is rationally bounded by L and it is such that

$$\chi_-(F' \cap M_L) \leqslant \chi_-(F)$$

and the surface $F' \cap M_L$ represents the same relative homology class of F.

Proof. First, we observe that trivial properly embedded disks in M_L, which are connected components of F, do not increase the complexity $\chi_-(F)$; then we can just delete them. Moreover, suppose that there are other boundary components of F, whose algebraic intersection with μ_i is zero for every $i = 1, ..., n$. Then either those components are parallel to the meridians, but this cannot happen because $F \cdot \mu_i = t$ for every i, either they are homologically trivial in the tori $\partial\nu(L)$, which means that they are circles which bound a disk. We can then push these disks slightly out of $\partial\nu(L)$, starting from the innermost ones, and then cap off the surface. After removing all the closed components we obtain a new surface whose complexity is smaller or equal to the one of F.

In the second step we show that we can take F' such that $F' \cap \partial\nu(L)$ consists of essential, parallel, simple closed curves all oriented in the same direction. Therefore, suppose there are two components C_1 and C_2 of $F' \cap \partial\nu(L_i)$ with opposite orientations; then there is an innermost pair of such components, say precisely C_1 and C_2 without loss of generality, such that $C_1 \cup C_2$ is the boundary of an annulus $A \subset \partial\nu(L_i)$ with $C_i \cap \mathring{A}$ empty. We can alter the surface F' by attaching a copy of the annulus A to F' and pushing it into the interior of M_L. This operation does not change the relative homology class of F' nor its complexity.

We have now obtained that F' is a compact, oriented surface properly embedded in M_L, representing the same relative homology class of F, such that $\chi_-(F') \leqslant \chi_-(F)$ and $F' \cap \partial\nu(L_i)$ is a link with slope (t, s), where $s \in \mathbb{Z}$, in the torus $\partial\nu(L_i)$ for every $i = 1, ..., n$. To conclude we need to extend F' inside $\nu(L_i)$ in a way that it is rationally bounded by L. This can always be done and it is proved in [2]. This completes the proof. □

From Lemma 2.1.3 and [86] we have that rational Seifert surfaces exist for every link in a rational homology sphere. Hence, we say that the *Thurston norm* of a link L with order t in M is the rational number

$$\|L\|_T = \left\| \frac{[F]}{t} \right\|_{M_L},$$

where F is a rational Seifert surface for L. In the same way as for null-homologous links, the following lemma implies that $\|L\|_T$ is well-defined.

Lemma 2.1.4 *In a rational homology sphere M, all the rational Seifert surfaces for an n-component link L, with order t in M, represent the same relative homology class in M_L.*

Proof. We reason in the same way as in the proof of Lemma 2.1.1. In fact, we observe that the intersection of each rational Seifert surface for L with M_L is clearly a compact, oriented and properly embedded surface F; but more importantly, every such surface is such that $F \cdot \mu_i = t$ whenever $i = 1, ..., n$. Then the claim follows again from Poincaré-Lefschetz duality. □

We also notice that if L is a null-homologous link then the two definitions coincide. In fact, in this case it is $t = 1$ and rational Seifert surfaces correspond precisely to genuine Seifert surfaces.

2.2 Results

In Subsection 1.1.1 we define the disjoint union of two links. We now show that the Thurston norm is additive under disjoint unions. First, we need the following definition.

A component K of a link L in a rational homology sphere M is called *compressible* if K is rationally bounded by an F, disjoint from $L \setminus K$, that is the image of a map $j : D \hookrightarrow M$, where D is diffeomorphic to a disk. We prove that compressible knots do not appear in every 3-manifold; more specifically, we show that we can find them only when M has a lens space as a connect summand.

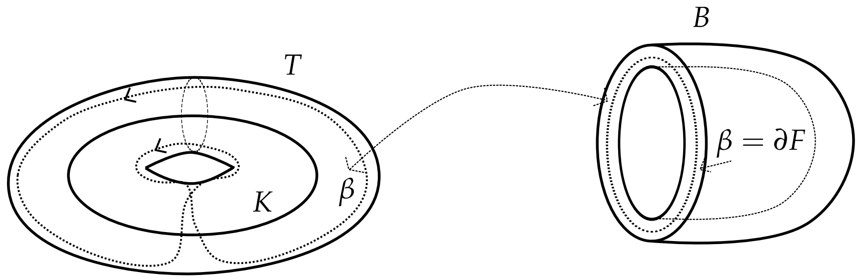


Figure 2.1: The ball B is attached to the solid torus T along the curve β.

Lemma 2.2.1 *Let us consider K, L and M as before and denote with L' the link $L \setminus K$. Suppose that K rationally bounds F, where F is determined by $j : D \to M$ with D a disk and $j|_D$ a p-fold cover of K. Moreover, suppose also that F is disjoint from L'. Then K is an order p knot in a lens space $L(p,q)$ such that $M = L(p,q) \# M'$ and $L' \hookrightarrow M'$.*

Proof. Let $T = \nu(K)$ and B be the tubular neighborhoods of K and $F \setminus \mathring{\nu}(K)$ respectively. Clearly, T is a solid torus and B is a 3-ball; moreover, we can suppose that $T \cup B$ is disjoint from L'.

Denote with β the simple, closed curve in the torus ∂T given by $\partial T \cap F$. Then we can see β as the attaching sphere of the 3-dimensional 2-handle B on T, which in turn can be thought as a lower handlebody. This implies that $T \cup B$ is diffeomorphic to a lens space minus a 3-ball. See Figure 2.1. Those are standard results in the theory of 3-manifolds and more details are in [41].

In particular, we have that $T \cup B \cong L(p,q) \setminus \{\text{3-ball}\}$ for some integer q, such that $\gcd(p,q) = 1$, and where p appears because $j|_D$ is a p-fold cover of K. Hence, it is $M = L(p,q) \# M'$ and the link L' is contained in M'. The fact that K is an order p knot in $L(p,q)$ is also straightforward, because K is the core of the handlebody T and it is easy to check that, for this reason, its homology class is the generator of $H_1(L(p,q);\mathbb{Z})$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. $\square$

We recall that a surface S is compressible in Y if a non-trivial circle in S bounds a disk in $Y \setminus S$. Hence, we can also say that K is compressible if it is embedded in a 3-manifold N, which is a connect summand of M, and $\partial\nu(K)$ is a compressible torus in N. Later in this section, we prove that these two definitions are equivalent.

Proposition 2.2.2 *Let us consider a link L in M such that $L = L_1 \sqcup L_2$, where each L_i is embedded in M_i with $M = M_1 \# M_2$. Then we have that*

$$\|L\|_T = \|L_1\|_T + \|L_2\|_T$$

and the order of L in M is $t = \mathrm{lcm}(t_1, t_2)$, where t_i is the order of L_i in M_i for $i = 1, 2$.

Proof. We suppose first that L_1 and L_2 have no compressible components. Let us consider a surface F which is rationally bounded by L. Fix a separating 2-sphere $S \hookrightarrow M$, that gives a connected sum decomposition of M, in a way that S intersects F transversely in a collection of circles $\mathcal{S}$. Moreover, we suppose that there exist neighborhoods for L_1 and L_2 which are disjoint from S, each one lying in one component of $M \setminus S$. Take the map $j : \Sigma \hookrightarrow M$ which defines F; if the neighborhoods are chosen small enough then we can also suppose that they intersect F only in the image of a neighborhood of $\partial\Sigma$.

Now each circle in $\mathcal{S}$ separates S into two disks. Let $C \subset \mathcal{S}$ be a circle that is innermost on F. This means that C bounds a disk D in S, the interior of which misses F. Now use D to do surgery on F in the following way: create a new surface $\widehat{F}$ from F by deleting a small annular neighborhood of C and replacing it by two disks, each a "parallel" copy of D, one on either side of D. We then perform the surgery on F described before on all the circles in $\mathcal{S}$. At this point $\widehat{F}$ may have closed components, but in this case we just delete them. Therefore, we are left with a disconnected surface whose connected components, that are no longer closed, stay in M_i according wether they bound a component of L_i. We call F_1 the surface given by the union of the components of $\widehat{F}$ of the first type and F_2 the other one.

We clearly have that F_1 and F_2 are disjoint, they lie in M_1 and M_2 respectively and that they do not intersect S; moreover, they are such that

- F_1 and F_2 have no closed components;
- $M_L \cap F_i$ is a proper submanifold of M_L with no disk components, while F_i coincide with F in a small neighborhood of L for $i = 1, 2$;
- $\chi(F_1) + \chi(F_2) = \chi(F_1 \sqcup F_2) \geqslant \chi(F)$.

Since F is rationally bounded by L and $F_1 \sqcup F_2$ coincide with F in a neighborhood of L, we have that F_i has L_i as boundary and then $[L_i] = [0]$ in the group $H_*(M; \mathbb{Q})$. This implies that t is a multiple of both t_1 and t_2. This proves that $t = \mathrm{lcm}(t_1, t_2)$.

We now want to show that the Thurston norm is additive. The fact that $\|L_1 \sqcup L_2\|_T \leqslant \|L_1\|_T + \|L_2\|_T$ follows immediately from Property 2 in Section 2.1. Then we suppose that the inequality is strict: this means that

$$-\frac{\chi(F_1)}{t} - \frac{\chi(F_2)}{t} \leqslant -\frac{\chi(F)}{t} < \|L_1\|_T + \|L_2\|_T \ .$$

In particular, at least one of the two surfaces, say F_1, is such that $-t^{-1}\chi(F_1) < \|L_1\|_T$. Then the claim follows from the fact that by construction, if $t = at_1$, the class $[F_1]$ coincide with a times the class represented by a rational Seifert surface of L_1 in $H_2(M_L, \partial M_L; \mathbb{Q})$; which gives a contradiction.

To conclude we need to prove that, if L is the disjoint union of a link L' with a compressible knot K, it is $\|L\|_T = \|L'\|_T + \|K\|_T$. This is done in the same way as the previous case, but we take into account the fact that disks do not increase the complexity of a surface. □

The proof of this proposition also implies the following corollary.

Corollary 2.2.3 *The Thurston norm of a link L in M_1 coincides with the one obtained when L is seen as a link in M, where $M = M_1 \# M_2$. Furthermore, the order of L in M_1 concides with the order of L in M.*

In light of Corollary 2.2.3, we use the symbol $\|L\|_T$ not only when L is seen as a link in M, but also in the case when L lies inside a connect summand of M. Moreover, we can now prove the following proposition.

Proposition 2.2.4 *The component K of a link L in M is compressible if and only if $L = K \sqcup L'$, where $K \hookrightarrow N$ with $M = N \# M'$, and the boundary of the tubular neighborhood of K is a compressible torus in N.*

Proof. Let us start to prove the only if implication. It follows easily from the definition of compressibility and Corollary 2.2.3 that K rationally bounds F, the image of a disk D under the map j, in N; where here j is such that $j|_D$ is a t-fold cover of K. It remains to show that t is actually the order of K in N.

We move to the if implication for the moment. In this case we have to show that the knot K is embedded in a connect summand N of M and $L' = L \setminus K$ does not intersects N. Then we can conclude by reasoning as in the proof of Proposition 2.2.2.

Knowing this, if we use Lemma 2.2.1 then we immediately prove both the previous claims. This completes the proof. $\square$

We observe that if K is a null-homologous knot then being compressible is equivalent to say that K is an unknot disjoint from L'.

We define the *compressibility term* of a link L as the rational number

$$\mathfrak{o}(L) = \sum_{i=1}^{o(L)} \frac{1}{t_i},$$

where $o(L)$ is the number of compressible components in L and $t_1, ..., t_{o(L)}$ are the orders of such components. Then, in order to prove our main result in Chapter 3 we need another lemma.

Lemma 2.2.5 *Suppose that L is a non-split link in a rational homology 3-sphere M. Then we have that*

$$\|L\|_T - \mathfrak{o}(L) = \min\left\{\frac{-\chi(F)}{t}\right\},$$

where F is rationally bounded by L and t is the order of L in M.

Proof. If L is not a compressible knot then $\mathfrak{o}(L) = 0$, because otherwise L would be split from Proposition 2.2.4, and so the claim follows from the definition of the Thurston norm and Lemma 2.1.3.

On the other hand, if the boundary of a tubular neighborhood of L is a compressible torus in M then F is rationally bounded by L, where F is the image of a map $j : D \hookrightarrow M$ with D diffeomorphic to a disk, and $\mathfrak{o}(L) = t^{-1}$. Therefore, its Thurston norm $\|L\|_T$ is equal to zero and the equality in the statement is given by F, since $\chi(F) = 1$. $\square$

Chapter 3

Contact structures and Legendrian links

3.1 Contact structures

3.1.1 Type of structures and convex surfaces

We recall that a *contact structure* ξ on an oriented 3-manifold M is a 2-plane field on M such that $\xi = \text{Ker } \alpha$, where α is a 1-form on M and $\alpha \wedge d\alpha$ is a volume 3-form for M. We say that ξ is cooriented by the form α. Moreover, we call a *contact 3-manifold* a pair (M, ξ) as before.

We define two relations between contact 3-manifolds. The first has the name of *contactomorphism* and it is given as follows. Consider (M_1, ξ_1) and (M_2, ξ_2) as above; they are contactomorphic if there exists an $F : M_1 \to M_2$, which is a diffeomorphism and it is such that $F_*(\xi_1) = \xi_2$.

On the other hand, given a 3-manifold M, we say that (M, ξ_1) is *contact isotopic* to (M, ξ_2) if we can find a map $F : M \times I \to M$ that satisfies the following properties:

- $F(\cdot, t)$ is a contactomorphism for every $t \in I$;
- $F(\cdot, 0) = \text{Id}_M$;
- $F(\cdot, 1)$ sends ξ_1 into ξ_2.

We observe that contactomorphisms give a relation between different contact 3-manifolds, while contact isotopies only relate contact structures on a fixed 3-manifold. Furthermore, the following corollary follows immediately from the previous definitions.

Corollary 3.1.1 *Suppose ξ_1 and ξ_2 are contact isotopic structures on a 3-manifold M. Then (M, ξ_1) and (M, ξ_2) are contactomorphic.*

An embedded disk E, in a contact manifold (M, ξ), is an *overtwisted disk* if its boundary ∂E is such that $TE \subset \xi$ and $E \cdot e_\xi = 0$, where e_ξ is the contact framing given by ξ. See [27]. In Section 3.2, we see that these conditions are equivalent to say that E is bounded by a Legendrian unknot in M and $\text{tb}_E(\partial E) = 0$, where tb is called Thurston-Bennequin number and it is also defined later.

A contact structure ξ on a 3-manifold M is an *overtwisted structure* if (M, ξ) contains an overtwisted disk. While, we say that ξ is a *tight structure* if it is not overtwisted.

We now need to recall the definition of convex surface in a contact 3-manifold and dividing set. We use [27] as a reference. We start by saying that a surface S in (M, ξ) is called *convex* if there is a vector field v, transverse to S, whose flow preserves ξ; such a vector field is called a *contact vector field*. We have the following approximation result.

Proposition 3.1.2 *Every closed surface Σ can be smoothly perturbed into a convex surface. Furthermore, in the case Σ has the link $L = L_1 \cup ... \cup L_n$ as boundary, if L is such that $TL \subset \xi$ and $\Sigma \cdot L_{i,\xi} \leqslant 0$ for $i = 1, ..., n$, where $L_{i,\xi}$ is the contact framing of L_i, then Σ can be again smoothly perturbed into a convex surface, fixing the boundary.*

Proof. The closed case is done in [37], while the proof for surfaces with boundary can be found in [50]. □

If v is a contact vector field in (M, ξ) transverse to S then the set

$$\Gamma_S = \{x \in S \mid v(x) \in \xi_x\}$$

is a collection of embedded curves, and arcs if the surface has non-empty boundary, in S and is called the *dividing set* of S in (M, ξ).

We have that the relative homotopy type of Γ_S is invariant under contact isotopy. Moreover, if the knot K is a component of the boundary of S, and it is Legendrian, which means such that $TK \subset \xi$, then

$$S \cdot K_\xi = -\frac{1}{2} \cdot |K \cap \Gamma_S| \,,$$

where K_ξ is the contact framing of K. More details can be found in [65].

3.1.2 Classification theorems

In this subsection we briefly state the most important results about existence and uniqueness of tight and overtwisted structures on a given 3-manifold.

Let us start with the overtwisted case. The major theorems in this settings are due to Eliashberg [18, 22].

Theorem 3.1.3 (Eliashberg) *The homotopy classes of 2-plane fields on an oriented, closed 3-manifold M are in bijection with overtwisted contact structures on M up to isotopy.*

This means that every 2-plane field on M is homotopic to an overtwisted contact structure and that two overtwisted structures, which are homotopic as 2-plane fields, are actually contact isotopic. In particular, overtwisted structures exist for every M.

In the specific case of S^3, all of the overtwisted structures can be explicitly determined. Let us recall that there is an isomorphism $d_3 : \pi_3(S^2) \to \mathbb{Z}$ and a generator of $\pi_3(S^2)$ is the Hopf fibration, introduced in [48]. The map d_3 is called the *Hopf invariant*. Then the following result holds.

Corollary 3.1.4 *The space of overtwisted structures on S^3 is in bijection with $\mathbb{Z}$ and then such structures are classified by the Hopf invariant d_3.*

Proof. From Eliashberg's Theorem 3.1.3 we have that an overtwisted structure ξ on S^3 is determined by the homotopy type of ξ as 2-plane field.

Since contact structures are cooriented, there is a vector field v such that $v(x)$ is orthogonal to ξ_x for every $x \in S^3$ and it is positively oriented. This means that ξ induces a map from S^3 to S^2 and then its homotopy type is an element of $\pi_3(S^2)$. □

From now on, we denote the overtwisted structures on S^3 with ξ_n, where $d_3(\xi_n) = n$ and ξ_0 is homotopic to the Hopf fibration.

We now suppose that M is non-compact. In this case, an overtwisted contact structure is called *tight at infinity* if it is tight outside a compact set. Otherwise, we say that it is *overtwisted at infinity*. We state the version of Theorem 3.1.3 for non-compact 3-manifold.

Theorem 3.1.5 (Eliashberg) *The classification of contact structures overtwisted at infinity coincides with the homotopical classification of tangent 2-plane distributions.*

On the other hand, we do not have a complete classification of tight contact structures, but Ding and Geiges in [16] proved that there exists a prime decomposition under contact connected sum. We say that a contact manifold (Y, ζ) is prime if $(Y, \zeta) = (Y_1, \zeta_1) \# (Y_2, \zeta_2)$ implies that at least one of the two summands is contactomorphic to (S^3, ξ_{st}), which is the unique tight structure on S^3 [21].

Theorem 3.1.6 (Ding and Geiges) *Every tight contact 3-manifold (M, ξ), that is not (S^3, ξ_{st}), is contactomorphic to a connected sum*

$$(M_1, \xi_1) \# \dots \# (M_k, \xi_k)$$

of finitely many prime tight contact 3-manifolds. The summands are unique up to order and contactomorphism.

More details on the connected sum of contact 3-manifolds can be found in [13].

3.1.3 Fillings

We recall that a smooth 4-manifold is symplectic if it is equipped with a closed non-degenerate differential 2-form ω, called a symplectic form. A contact manifold (M, ξ) is called *weakly symplectically fillable* if there is a compact symplectic 4-manifold (X, ω) such that $\partial X = M$ and $\omega|_\xi > 0$.

We also say that (M, ξ) is *strongly symplectically filled by* X if there is a vector field v, transversely pointing out of X along M, such that the Lie derivative $L_v\omega$ is a positive multiple of ω in every point and the contraction $\iota_v\omega$ is a contact 1-form for ξ.

It is clear from the definition that a strong filling is also a weak filling. Moreover, the two notions coincide on rational homology spheres. See [20, 25, 63].

An important result of Eliashberg and Gromov is that simplectically fillable contact structures are always tight.

Theorem 3.1.7 (Eliashberg and Gromov) *Suppose that (M, ξ) is a contact 3-manifold which admits a weak symplectic filling. Then the structure ξ is tight.*

This theorem was reproved by Ozsváth and Szabó, using Heegaard Floer homology, in [70].

Another type of filling that is interesting to study consists of the family of Stein fillings. A *Stein manifold* is a triple (X, J, ψ), where J is a complex structure on X, there is a Morse function ψ from X to $\mathbb{R}$ and $\omega_\psi(v, w) = -\mathrm{d}(\mathrm{d}\psi \circ J)(v, w)$ is non-degenerate.

A contact manifold (M, ξ) is called *Stein fillable* if there is a Stein manifold (X, J, ψ) such that

- the map ψ is bounded from below;

- the manifold M is a non-critical level of ψ;
- $-(\mathrm{d}\psi \circ J)$ is a contact form for ξ.

The following corollary follows easily from the definition of Stein filling.

Corollary 3.1.8 *A Stein fillable contact 3-manifold is strongly symplectically fillable*

The converse of this result is not true as we know from Ghiggini [36].

We can now say a few words on contact surgeries. More details are in [27]. Let K be a knot in a contact 3-manifold (M, ξ) such that $TK \subset \xi$ (a Legendrian knot). It is known, see [35], that K has a neighborhood $\nu(K)$ that is contactomorphic to a neighborhood of the x-axis in $(\mathbb{R}^3, \mathrm{Ker}\,(\mathrm{d}z - y\mathrm{d}x))/\sim$, where $\sim$ identifies (x, y, z) with $(x + 1, y, z)$.

With respect to these coordinates on $\nu(K)$, we can remove $\nu(K)$ from M and topologically glue it back by performing a Dehn surgery along K, with framing the contact framing ± 1. Denote the resulting manifold with $M_{(K,\pm 1)}$. There is a unique way, up to isotopy, to extend $\xi|_{M \setminus \nu(K)}$ to a contact structure $\xi_{(K,\pm 1)}$ over all of $M_{(K,\pm 1)}$ so that $\xi_{(K,\pm 1)}|_{\nu(K)}$ is tight, see [46]. The new contact manifold $(M, \xi)_{(K,\pm 1)}$ is said to be obtained from (M, ξ) by ± 1-*contact surgery* along K.

We can iterate this procedure multiple times on different Legendrian knots in M. Then in this case we say that we are doing a contact surgery on (M, ξ) along a Legendrian link. We recall that every (M, ξ) is obtained from (S^3, ξ_{st}) by contact surgery [15]. We conclude this subsection with the following theorem.

Theorem 3.1.9 *Suppose that a contact manifold (M, ξ) admits a contact surgery presentation which contains only -1-surgeries. Then (M, ξ) is Stein fillable.*

This fact is proved in [19, 40].

3.2 Legendrian links

A link L in a contact 3-manifold (M, ξ) is called *Legendrian* if $TL \subset \xi$, which means that $T_pL \subset \xi_p$ for every $p \in L$.

Two Legendrian links L_1 and L_2 are *Legendrian isotopic* if there is a contact isotopy $G : M \times I \to M$ that sends L_1 into L_2.

A number or an element of a group is a Legendrian invariant if it remains unchanged under Legendrian isotopy. In the following thesis, we call classical invariants of Legendrian links the link type, the Thurston-Bennequin number and the rotation number. The definitions of these invariants, that are given for links in this section, coincide with the ones in [2] for knots.

The link type is the most obvious Legendrian invariant. In fact a Legendrian isotopy of links is also a smooth isotopy. The second one is the Thurston-Bennequin number, which we denote with tb and it is defined as follows. Let us consider a Legendrian link L, with non-split link type for now, in a rational homology contact 3-sphere (M, ξ). If we call $N(L)$ the normal bundle over L then we have that, since L is Legendrian, $N(L) \cap \xi$ determines a non-zero vector field v along L, which is uniquely determined after normalization. This is the definition of the contact framing of L in (M, ξ).

Let L_ξ be the link obtained by pushing L along the contact framing. We extract a rational number from L_ξ by saying that

$$\mathrm{tb}(L) = \mathrm{lk}_{\mathbb{Q}}(L, L_\xi) = \frac{F \cdot L_\xi}{t} \;;$$

where F is a rational Seifert surface for L in M, as defined in Section 2.1, and t is the order of L in M.

We define the *Thurston-Bennequin number* of every Legendrian link L by taking $\mathrm{tb}(L) = \mathrm{tb}(L_1) + ... + \mathrm{tb}(L_r)$, where the L_i's correspond to the split components of the link type of L. It follows from the properties of the linking number in Subsection 1.1.1 that $\mathrm{tb}(L)$ does not depend on the choice of the rational Seifert surface. Moreover, it is also easy to check that this number is a Legendrian invariant.

The Thurston-Bennequin number satisfies the properties stated in the following proposition.

Proposition 3.2.1 *Suppose that L is a Legendrian link in a rational homology contact 3-sphere (M,ξ). Then we have that*

1. *if L is null-homologous then* $\mathrm{tb}(L)$ *is an integer;*
2. *if we reverse the orientation of L then* $\mathrm{tb}(-L) = \mathrm{tb}(L)$.

Proof. The link L is null-homologous in M if and only if L_i is null-homologous in M for every $i = 1,...,r$, where the L_i's correspond to the split components of the link type as before, because of Corollary 2.2.3. Then the first claim follows from Proposition 1.1.2.

Now we observe that $-F$ represents a rational Seifert surface for $-L$ and $-L_\xi$ is its contact framing. Then this clearly means that nothing changes in the definition of $\mathrm{tb}(-L)$. □

We note that, when M is not a rational homology sphere, the Thurston-Bennequin number may depend on the choice of the surface F, even if L is null-homologous; in particular, it depends on the relative homology class of the surface we take. Therefore, in this case we define the invariant as

$$\mathrm{tb}_\Sigma(L) = F \cdot L_\xi \,,$$

where F is a Seifert surface for L representing the class $\Sigma \in H_2(M, M_L; \mathbb{Z})$.

The third classical invariant of L is the rotation number $\mathrm{rot}(L)$. As before, we take M as a rational homology sphere. Let F be a rational Seifert surface for L. Recall that we have an inclusion $j : \Sigma \hookrightarrow M$, which is an embedding on the interior of Σ and an r-fold cover $\partial\Sigma \to L$. Then we can trivialize the pull-back $i^*\xi$ on Σ, because a surface with boundary retracts on a bouquet of circles.

This means that there exist two sections $s_1, s_2 : \Sigma \to i^*\xi$ such that $\{s_1(p), s_2(p)\}$ is a basis of $(i^*\xi)_p$ for every $p \in \partial\Sigma$. Therefore, if u is the unit vector field tangent to L and oriented coherently, we have that $(i^*u)_p \in (i^*\xi)_p$, since L is Legendrian, and

$$(i^*u)_p = \alpha(p)s_1(p) + \beta(p)s_2(p) \quad \text{for any } p \in \partial\Sigma \,;$$

where α and β are real-valued functions.

From this we define a map γ in the following way

$$\gamma : \partial\Sigma \longrightarrow S^1 \sqcup ... \sqcup S^1$$

$$p \longmapsto \frac{(\alpha(p), \beta(p))}{||\,(\alpha(p), \beta(p))\,||}$$

and we call *rotation number* of L the rational number

$$\mathrm{rot}(K) = \frac{1}{t} \cdot \sum_i \deg(\gamma_i) \,,$$

where γ_i is the restriction of γ to the i-th component of $\partial\Sigma$ and t is the order of L in M. In the same way as before, the value of $\operatorname{rot} L$ does not depend on the choice of F nor of the s_i's and it is a Legendrian invariant.

The following proposition is the version of Proposition 3.2.1 for $\operatorname{rot} L$.

Proposition 3.2.2 *Suppose that L is a Legendrian link in a rational homology contact 3-sphere (M, ξ). Then we have that*

1. *if L is null-homologous then* $\operatorname{rot}(L)$ *is an integer;*
2. *if we reverse the orientation of L then* $\operatorname{rot}(-L) = -\operatorname{rot}(L)$.

Proof. Property 1 is trivial. In order to prove Property 2, we take $-F$ as rational Seifert surface and $-u$ as unit tangent vector field, but we can still take s_1 and s_2 as sections. This means that, if γ' is the map for $-L$, it is $\deg(\gamma_i') = -\deg(\gamma_i)$ for every i. □

3.3 Transverse links

A smooth link $T \hookrightarrow (M, \xi)$ is called *transverse* if $T_pT \oplus \xi_p = T_pM$ for every $p \in T$ and $\alpha|_T$ is a volume 1-form for T. This definition implies that transverse links always come with a natural orientation, induced by ξ.

As for Legendrian links, we say that two transverse links T_1 and T_2 are *transverse isotopic* if there is a contact isotopy $G : M \times I \to M$ that sends T_1 into T_2.

A Legendrian link $L \hookrightarrow (M, \xi)$ always determines two special transverse links named positive and negative *transverse push-off* of L, that we denote with T_L^+ and $-T_L^-$, where the minus sign appears because transverse links need to be oriented accordingly to the contact structure.

We briefly recall the construction of $T_L^{\pm}$. Let $A_i = S^1 \times [-1, 1]$ be a collection of embedded annuli in M such that $S^1 \times \{0\} = L_i$ and A_i is transverse to ξ for every $i = 1, ..., n$, where n is the number of components of L. If the A_i's are sufficiently thin, see [26], then $T_L^{\pm}$ is the link with components $S^1 \times \{\pm\frac{1}{2}\}$, which is transverse up to orientation. It is easy to check that any two positive (or negative) transverse push-offs are transversely isotopic and then $T_L^{\pm}$ is uniquely defined, up to transverse isotopy. Moreover, if we reverse the orientation of L then we have that $T_{-L}^{\pm} = -T_L^{\mp}$.

Since we are working with rational homology spheres, we can define the self-linking number $\operatorname{sl}(T)$ of a transverse link T with order t in (M, ξ). The self-linking number is one of the two classical invariants of transverse links, together with the link type. Suppose for now that T is non-split; take a surface F which is rationally bounded by T and the map $j : \Sigma \to M$ that defines F, as described before in Section 2.1. Consider the pull-back bundle $j^*\xi$ on Σ. Then again we have that $j^*\xi$ is trivial. Let v be a non-zero section of $j^*\xi$. Normalize v so that $v|_{\partial\Sigma}$ defines a link T_ξ in $\partial\nu(T)$. We define the *self-linking number* of T to be

$$\operatorname{sl}(T) = \frac{1}{t} \cdot \operatorname{lk}_{\mathbb{Q}}(T, T_\xi) = \frac{F \cdot T_\xi}{t^2} .$$

This definition coincides with the one given by Baker and Etnyre in [2] for transverse knots.

In the case that T has split link smooth type, we consider $T_1, ..., T_r$ the connected components of T as a smooth link. Then we say that

$$\operatorname{sl}(T) = \sum_{i=1}^{r} \operatorname{sl}(T_i) .$$

The fact that M is a rational homology sphere tells us that $\mathrm{sl}(T)$ is independent of the choice of the surface F. Furthermore, the self-linking number is a transverse invariant and, as in the Legendrian case, it has the following property.

Corollary 3.3.1 *Suppose that T is a null-homologous transverse link in a rational homology contact 3-sphere (M,ξ). Then the invariant $\mathrm{sl}(T)$ is an integer.*

We can find the value of the self-linking number of the transverse push-offs of a Legendrian link L, starting from its Thurston-Bennequin and rotation numbers.

Proposition 3.3.2 *Let L be an n-component Legendrian link in the rational homology contact 3-sphere (M,ξ). Then we have that*

$$\mathrm{sl}\left(T^+_{\pm L}\right) = \mathrm{tb}(L) \mp \mathrm{rot}(L)\,.$$

Proof. The proof is done in [2] for Legendrian knots. Hence, the claim follows by observing that

$$\begin{aligned}\mathrm{tb}(L) \mp \mathrm{rot}(L) &= \sum_{i=1}^{n} \mathrm{tb}(L_i) + 2\sum_{i\neq j} \mathrm{lk}_{\mathbb{Q}}(L_i,L_j) \mp \sum_{i=1}^{n} \mathrm{rot}(L_i) = \\ &= \sum_{i=1}^{n} \mathrm{sl}(T^+_{\pm L_i}) + 2\sum_{i\neq j} \mathrm{lk}_{\mathbb{Q}}(L_i,L_j) = \mathrm{sl}(T^+_{\pm L})\,,\end{aligned}$$

where the first and the last equality are consequences of Equation (1.1). □

3.4 Legendrian and transverse links in (S^3,ξ_{st})

3.4.1 Front projections of Legendrian links

In Subsection 1.2.1 we saw that smooth links in S^3 can be represented by planar diagrams; moreover, two such diagrams, corresponding to isotopic links, differ by a sequence of Reidemeister moves (Figure 1.5). We now describe a similar construction for Legendrian links in (S^3,ξ_{st}).

Since a link in S^3 is embedded in $\mathbb{R}^3$ and $\xi_{\mathrm{st}}|_{\mathbb{R}^3}$ is still tight, we can work in $(\mathbb{R}^3,\xi)$, where $\xi = \mathrm{Ker}(\mathrm{d}z - y\mathrm{d}x)$. In fact, there is a unique tight contact structure on $\mathbb{R}^3$ up to contact isotopy [21]. Hence, we can define the *front projection* of an n-component Legendrian link L. This is the map

$$\begin{aligned}\bigsqcup_{i=1}^{n} S^1 &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto (x(\theta),z(\theta))\end{aligned}$$

where $\theta = (\theta_1,...,\theta_n)$ and $(x(\theta),y(\theta),z(\theta))$ is the parametrization of L.

A generic front projection of L is a special diagram for L with no vertical tangencies, that are replaced by cusps, and at each crossing, the slope of the overcrossing is smaller than the one of the undercrossing. An example of front projection is shown in Figure 3.1: we picture a diagram of the *standard Legendrian unknot $\mathcal{O}$*, which is the unique, up to Legendrian isotopy, Legendrian unknot such that $\mathrm{tb}(\mathcal{O}) = -1$. This is proved in [24].

There exist moves on Legendrian front projections, called the *Legendrian Reidemeister moves* and they are illustrated in Figure 3.2. A further move is the Legendrian planar isotopy; such a move is an isotopy of the front projection that does not introduce vertical tangencies. These moves appear in the following Legendrian analogue of Reidemeister's Theorem 1.2.1. A proof can be found in [85].

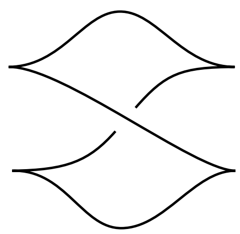
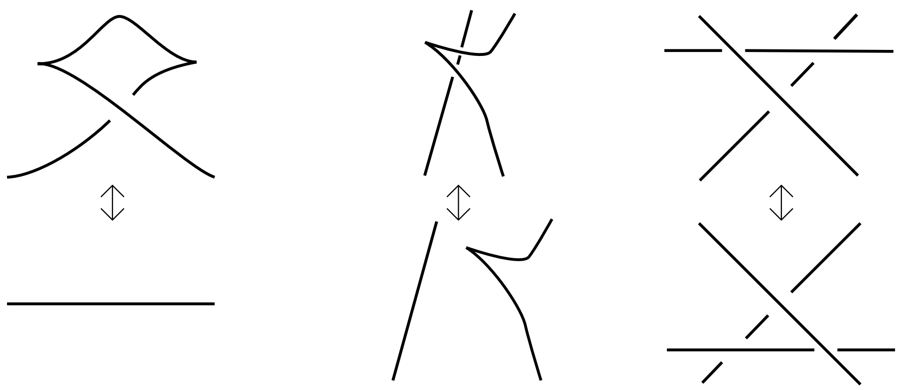


Figure 3.1: A front projection of the standard Legendrian unknot.

Theorem 3.4.1 (Świątkowski) *Two front projections correspond to Legendrian isotopic Legendrian links if and only if the projections can be connected by Legendrian planar isotopies and by Legendrian Reidemeister moves.*

The classical invariants of an oriented Legendrian link L can be easily computed from its oriented front projection P. We recall that we define the writhe in Subsection 1.1.1. Then we have the following formulas:



Type 1 Type 2 Type 3

Figure 3.2: The three Legendrian Reidemeister moves. For each move, we consider all the possible reflections and orientations, provided that they satisfy the conditions in the definition of front projection.

$$\mathrm{tb}(L) = \mathrm{wr}(P) - \frac{1}{2} \cdot |\text{cusps in } P|\,; \tag{3.1}$$

$$\mathrm{rot}(L) = \frac{1}{2} \cdot \Big(|\text{down-ward cusps in } P| - |\text{up-ward cusps in } P| \Big)\,. \tag{3.2}$$

3.4.2 Braid presentation for transverse links

Transverse links can be studied by using closed braids. We recall that a *closed braid* is an n-component link in $\mathbb{R}^3$, where here we use cylindrical coordinates (r, ψ, z), that can be parametrized by a map

$$\begin{aligned} \bigsqcup_{i=1}^{n} S^1 &\longrightarrow \mathbb{R}^3 \\ \theta &\longmapsto (r(\theta), \psi(\theta), z(\theta)) \end{aligned}$$

for which $\theta = (\theta_1, ..., \theta_n)$, $r(\theta) \neq 0$ and $\psi'(\theta) > 0$ for every θ. More details on braids and closed braids can be found in [5].

To see the connection between braids and transverse links we use the contact structure ξ_{sym} on $\mathbb{R}^3$, which is given by $\xi_{\mathrm{sym}} = \mathrm{Ker}\,(\mathrm{d}z + r^2 \mathrm{d}\psi)$. Since this structure is contact isotopic to the standard one, we can transfer any question about ξ_{st} to ξ_{sym}. Now given a closed braid B, we can isotopy it through closed braid so that it is enough far from the z-axis that the planes that make up ξ_{sym} are almost vertical, which means they are close to the planes spanned by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial r}$. Thus the closed braid type B represents a transverse link. Bennequin proved the opposite assertion in [4].

Theorem 3.4.2 (Bennequin) *Every transverse link in* $(\mathbb{R}^3, \xi_{sym})$ *is transversely isotopic to a closed braid.*

Proof. The proof of the theorem follows the same strategy of the one that smooth links can be braided. See [5]. We just need to check that such procedure can be done in a transverse way. For details we refer to [4, 64]. □

Let us fix k points p_i in a disk D^2, for $i = 1, ..., k$. Then we recall that a *k-braid* is an embedding of k arcs $\gamma_i : [0,1] \to D^2 \times [0,1]$ so that $\gamma_i(t) \in D^2 \times \{t\}$ and the endpoints of the γ_i's, corresponding to 0 (or 1), map as a set to p_i in $D^2 \times \{0\}$ (or $D^2 \times \{1\}$). It is easy to

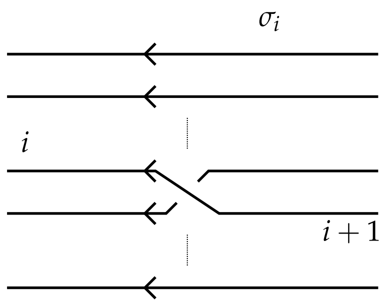


Figure 3.3: The generator σ_i for the braid group B_k.

see that braids and closed braids are in one-to-one correspondence.

The set of all the k-braids B_k forms a group. Such group is generated by σ_i for $i = 1, ..., k-1$, where σ_i is as in Figure 3.3. Given a braid b in B_k, then the positive stabilization of b is $b\sigma_k$ in B_{k+1}. Moreover, a conjugation of b is $\beta b \beta^{-1}$ for some $\beta \in B_k$. Then from [64, 91] we have that braid presentations of transversely isotopic links are related by those two moves.

Theorem 3.4.3 (Orevkov and Shevchishin; Wrinkle) *Two braids represent the same transverse link if and only if they are related by a finite sequence of positive stabilizations (and their inverse) and conjugations in the braid group.*

From [4] we also have that the self-linking number of a transverse link T in (S^3, ξ_{st}), represented by the closed k-braid B, is computed as follows:

$$\mathrm{sl}(T) = \mathrm{wr}(B) - k\,, \tag{3.3}$$

where $\mathrm{wr}(B)$ is the writhe of B seen as a link diagram for T.

3.5 Connected sum and disjoint union in the contact setting

The definition of connected sum of two 3-manifolds can be easily given also when they come together with a contact structure. Let us take two connected contact manifolds (M_1, ξ_1) and (M_2, ξ_2); we call M_i' (for $i = 1, 2$) the manifolds obtained from M_i by removing an open *Darboux ball*, that is a 3-ball with the its unique tight contact structure [21], and we define $(M_1 \# M_2, \xi_1 \# \xi_2)$ the contact manifold which is gotten by gluing together M_1' and M_2'. The structure $\xi_1 \# \xi_2$ is well-defined because it is always possible to glue two contact structures along the boundary of a Darboux ball. Moreover, the result is independent of the choice of the balls themselves and we have the following proposition.

Proposition 3.5.1 *For every Darboux ball B in the overtwisted manifold (M, ξ) there exists at least one overtwisted disk disjoint from B. In particular, if $(M, \xi) = (M_1 \# M_2, \xi_1 \# \xi_2)$ and a summand M_i is overtwisted then we can always find overtwisted disks which are contained entirely in $M_i' \subset M_1 \# M_2$.*

Proof. From Eliashberg's classification, the connected sum of (M, ξ) with (S^3, ξ_0) is always contact isotopic to (M, ξ). Then the first statement can be proved by noting that we can find two disjoint Darboux balls in (M, ξ) and, since ξ is overtwisted, one can be replaced with (S^3, ξ_0) minus a Darboux ball. At this point, the second statement is an immediate consequence of the first one. □

Clearly if (M_1, ξ_1) is overtwisted then the connected sum is still overtwisted, but if both the summands are tight then it is important to cite the following result of Colin [13].

Theorem 3.5.2 *The connected sum of two contact 3-manifolds $(M_1 \# M_2, \xi_1 \# \xi_2)$ is tight if and only if (M_1, ξ_1) and (M_2, ξ_2) are both tight.*

If a sphere S separates a contact manifold (M, ξ) into two components M_1 and M_2 such that smoothly it is $M = M_1 \# M_2$ then we can ask whether ξ also splits accordingly. Then we can state the following lemma.

Lemma 3.5.3 *A convex separating sphere S in (M, ξ) gives a connected sum decomposition $(M, \xi) = (M_1 \#_S M_2, \xi_1 \#_S \xi_2)$ if and only if the dividing set Γ_S is trivial, which means that consists of a unique simple, closed curve.*

Proof. The claim follows from the fact that a contact manifold, whose boundary is a convex sphere, can be glued together with a Darboux ball if and only if its dividing set is trivial as shown in [27]. □

Etnyre and Honda in [31] extended the definition of connected sum to Legendrian links. Suppose that we have two Legendrian links L_1 and L_2 in (M_1, ξ_1) and (M_2, ξ_2) respectively. We take two Darboux balls D_1 and D_2 as before, but with the condition that $D_i \cap L_i$ is a Legendrian arc α_i where $\alpha_i \cap \partial D_i$ consists of two points and its front projection is isotopic to the one in Figure 3.4; we call this the *Legendrian elementary tangle*. In this way, the link $L_1 \# L_2$ in $(M_1 \# M_2, \xi_1 \# \xi_2)$ is Legendrian and it does not depend on the choice of the Darboux balls D_i, but only of which component of L_i contains the arc α_i.

We observe that we have:

- $\mathrm{tb}(L_1 \# L_2) = \mathrm{tb}(L_1) + \mathrm{tb}(L_2) + 1$;
- $\mathrm{rot}(L_1 \# L_2) = \mathrm{rot}(L_1) + \mathrm{rot}(L_2)$.

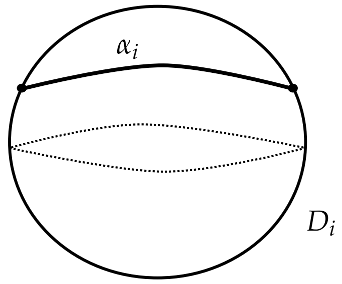


Figure 3.4: Front projection of α_i in a Darboux ball.

We define positive (negative) *stabilization* of a Legendrian link L in (S^3, ξ_{st}), with front projection P, the Legendrian link $L^\pm$ represented by the front projection $P^\pm$; which is obtained by adding two consecutive down-ward (up-ward) cusps to P. Stabilizations are well-defined, in the sense that they do not depend on the choice of the point of P where we add the new cusps.

At this point it is easy to define stabilizations in every contact manifold. In fact we say that $L^\pm$ is the positive (negative) stabilization of L, a Legendrian link in (M, ξ), if $L^\pm = L \# \mathcal{O}^\pm$; where $\mathcal{O}^\pm$ is the positive (negative) stabilization of the standard Legendrian unknot $\mathcal{O}$ in (S^3, ξ_{st}). The Legendrian knots $\mathcal{O}^\pm$ are shown as front projections in Figure

Figure 3.5: Front projection of $\mathcal{O}^+$ (left) and $\mathcal{O}^-$ (right).

3.5. From Equations (3.1) and (3.2) it results that:

- $\text{tb}(L^\pm) = \text{tb}(L) - 1$;
- $\text{rot}(L^\pm) = \text{rot}(L) \pm 1$.

Now, consider L_i Legendrian links in (M_i, ξ_i) for $i = 1, 2$. We define the *disjoint union*

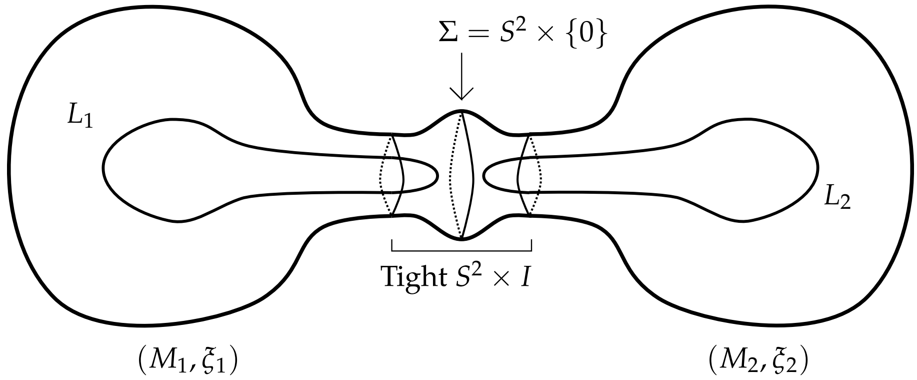


Figure 3.6: Disjoint union of L_1 and L_2.

$L_1 \sqcup L_2$ in $(M_1 \# M_2, \xi_1 \# \xi_2)$ as the Legendrian link given by a particular connected sum $L_1 \# \mathcal{O}_2 \# L_2$; where $\mathcal{O}_2$ is the standard Legendrian unlink with 2 components in (S^3, ξ_{st}). The connected sum is such that one component of $\mathcal{O}_2$ is summed to L_1 and the other one to L_2 as shown in Figure 3.6. Note that L_1 and L_2 are Legendrian links in $(M_1 \# M_2, \xi_1 \# \xi_2)$.

Using the same criterion of the smooth case in Subsection 1.1.1, we say that a Legendrian link L, in a contact 3-manifold (M, ξ), is *split* if there exist $L_i \hookrightarrow (M_i, \xi_i)$ for $i = 1, 2$ such that $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$ and L is Legendrian isotopic to the disjoint union of L_1 and L_2. Otherwise, we say that L is *non-split*.

Finally, we note that as before:

- $\text{tb}(L_1 \sqcup L_2) = \text{tb}(L_1) + \text{tb}(L_2)$;
- $\text{rot}(L_1 \sqcup L_2) = \text{rot}(L_1) + \text{rot}(L_2)$.

3.6 The Thurston-Bennequin inequality

The Thurston-Bennequin inequality is a very powerful result in contact topology. One of its most important implications is that, provided the structure is tight, the Thurston-Bennequin numbers of all the Legendrian knots, with the same smooth knot type, are bounded from above. This property actually characterizes tight contact structures on 3-manifolds; in fact, when the structure is overtwisted, it is always possible to increase the Thurston-Bennequin number indefinitely, without changing the smooth isotopy class of the knot. This is done by connect summing with the boundary of an overtwisted disk.

A proof for a version of the Thurston-Bennequin inequality was given by Eliashberg in [23]. In the following thesis we show that an analogous inequality, involving the Thurston norm $\|L\|_T$ of a link defined in Chapter 2, holds for Legendrian and transverse links in every rational homology contact 3-sphere, equipped with a tight structure. For Legendrian links in (S^3, ξ_{st}) this result, and the resulting upper bound for the maximal Thurston-Bennequin number, was found first by Dasbach and Mangum in [14]. On the other hand, if we consider only Legendrian and transverse knots then our bounds coincide with the ones of Baker and Etnyre [2].

We recall that if Σ is a surface in (M, ξ) then, at each point $x \in \Sigma$, we have that $l_x = \xi_x \cap T_x\Sigma$ is a singular line field on Σ. This line field can be integrated to give a singular foliation on Σ. This singular foliation is called the *characteristic foliation* and is denoted with Σ_ξ. More details on foliations are found in [90].

In order to prove our result, we use the same strategy of [2, 23] and then the proof follows from the following lemma.

Lemma 3.6.1 *Suppose that T is a non-split transverse link in a rational homology tight 3-sphere (M, ξ) and take an F in M that is rationally bounded by T. Then we can perturb F in a way that $T' = F \cap \partial\nu(T)$ is still transverse and* $\text{sl}(T') = t \cdot \text{sl}(T)$.

Furthermore, we have that

$$F \cdot T'_\xi \leqslant -\chi(F)\,,$$

where T'_ξ is the framing defined as in Section 3.3.

Proof. Consider the map $j : \Sigma \to M$ which determines F. Let us take a small neighborhood $\nu(T)$ of T such that it intersects F only in the image of a neighborhood of $\partial\Sigma$, as in the proof of Proposition 2.2.2. Denote the properly embedded surface $M_L \cap F$ with F', where the manifold M_L is $M \setminus \mathring{\nu}(T)$.

We know from [2] that we can modify F' in a way that $\partial F' = T'$ is as wanted and the characteristic foliation F'_ξ is generic. In particular, we can assume that all the singularities are isolated elliptic or hyperbolic points. Moreover, each singularity has a sign depending on whether the orientation of ξ and TF' agree at the singularity. Let us denote with $e_\pm$ and $h_\pm$ the number of such singularities.

Since we can interpret $F' \cdot T_\xi$ as a relative Euler class, in other words $F' \cdot T_\xi$ is the obstruction to extending the framing T_ξ to a non-zero vector field on F', we have that

$$F \cdot T'_\xi = F' \cdot T'_\xi = (e_- - h_-) - (e_+ - h_+)\,,$$

as in [2]. Moreover, a simple computation gives that

$$\chi(F) = (e_+ + e_-) - (h_+ + h_-)\,;$$

thus it is

$$F \cdot T'_\xi + \chi(F) = 2(e_- - h_-)\,. \tag{3.4}$$

At this point, since ξ is tight, every negative elliptic point is connected to a negative hyperbolic point, otherwise we could find overtwisted disks in (M, ξ), and then we can cancel this pair using Giroux's elimination lemma [38]. This means that we can isotope F' so that the characteristic foliation is such that $e_- = 0$ and then the claim follows easily from Equation (3.4). □

We can now prove the main result of the chapter.

Theorem 3.6.2 *Suppose that L is a Legendrian link in a tight contact 3-manifold (M, ξ) and M is a rational homology sphere. Then we have that*

$$\mathrm{tb}(L) + |\,\mathrm{rot}(L)| \leqslant \|L\|_T - \mathfrak{o}(L)\,.$$

Furthermore, suppose that T is a transverse link in (M, ξ). Then we have that

$$\mathrm{sl}(T) \leqslant \|T\|_T - \mathfrak{o}(T)\,,$$

where the Thurston norm and the compressibility term $\mathfrak{o}$ are defined in Chapter 2.

Proof. Suppose first that T is a non-split transverse link of order t in (M, ξ). Take an F that is rationally bounded by T and gives the equality in Lemma 2.2.5. Then from Lemma 3.6.1 we have that

$$\mathrm{sl}(T) = \frac{\mathrm{sl}(T')}{t} = \frac{F \cdot T'_\xi}{t} \leqslant -\frac{\chi(F)}{t} = \|T\|_T - \mathfrak{o}(T)\,.$$

For the second equality we also use that T' is null-homologous.

On the other hand, if T has split smooth link type then we reason as in Section 3.3 and we write $T_1, ..., T_r$, which are the split components of T. Hence, this time we obtain

$$\mathrm{sl}(T) = \sum_{i=1}^{r} \mathrm{sl}(T_i) \leqslant \sum_{i=1}^{r} \|T_i\|_T - \mathfrak{o}(T_i) = \|T\|_T - \mathfrak{o}(T)\,,$$

because of Proposition 2.2.2 and the additivity of the Thurston norm and the compressibility term.

Now suppose that $L \hookrightarrow (M, \xi)$ is a Legendrian link. Then from Proposition 3.3.2 we have that

$$\mathrm{tb}(L) \mp \mathrm{rot}(L) = \mathrm{sl}\left(T^+_{\pm L}\right) \leqslant \|L\|_T - \mathfrak{o}(L)\,;$$

in fact, L and its transverse push-offs are all smoothly isotopic, up to orientation, and then they have same Thurston norm and number of disjoint unknots. □

Chapter 4

Representations of links and 3-manifolds

4.1 Heegaard diagrams

4.1.1 3-manifolds and Spin^c structures

In this chapter we describe two ways of presenting links in 3-manifolds. Heegaard diagrams are crucial for the construction of Heegaard Floer homology, both the version for manifolds than the one for links, while in Chapter 6 we use open book decompositions in order to define some contact and Legendrian invariants.

Hereby, we use terminology from [41, 67, 72]. An oriented *handlebody* is an oriented 3-manifold with boundary obtained by attaching 3-dimensional 1-handles to a 3-ball. It is a classical result on 3-manifolds that any connected, oriented, closed 3-manifold M can be decomposed, along a separating surface Σ, as the union of two handlebodies [84]. This is called a Heegaard decomposition of M.

Let Σ be an oriented, closed, genus g surface, which means a connected sum of g tori, and fix a g-tuple of homologically linearly independent, mutually disjoint curves $\gamma = \{\gamma_1, ..., \gamma_g\}$; this is equivalent to say that $\Sigma \setminus (\gamma_1 \cup ... \cup \gamma_g)$ is a connected, planar surface. We call such a collection of curves a *system of attaching circles*. These curves always specify

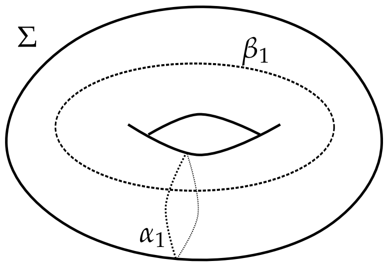


Figure 4.1: A Heegaard diagram for the 3-sphere. The torus Σ splits S^3 into two solid tori, which are precisely the handlebodies U_α and $-U_\beta$.

a handlebody with boundary Σ, uniquely up to diffeomorphism inducing the identity on the boundary; moreover, the γ_i's bound disjoint, embedded disks D_i in the handlebody.

A *Heegaard diagram* is a triple (Σ, α, β), consisting of an oriented surface Σ equipped with two systems of attaching circles $\alpha = \{\alpha_1, ..., \alpha_g\}$ and $\beta = \{\beta_1, ..., \beta_g\}$, which intersect transversely. The associated 3-manifold M and its Heegaard splitting consist of the handlebodies U_α and $-U_\beta$, specified by the g-tuples α and β, glued to Σ. See Figure 4.1.

We now define an enriched version of Heegaard diagrams. More specifically, a *pointed Heegaard diagram* is a quadruple $H = (\Sigma, \alpha, \beta, w)$ given by a triple (Σ, α, β) as before and a point $w \in \Sigma \setminus (\alpha \cup \beta)$.

Now, fix a connected, oriented 3-manifold M, and a generic self-indexing Morse function on M, which has one index zero and one index three critical point. Fix also a generic

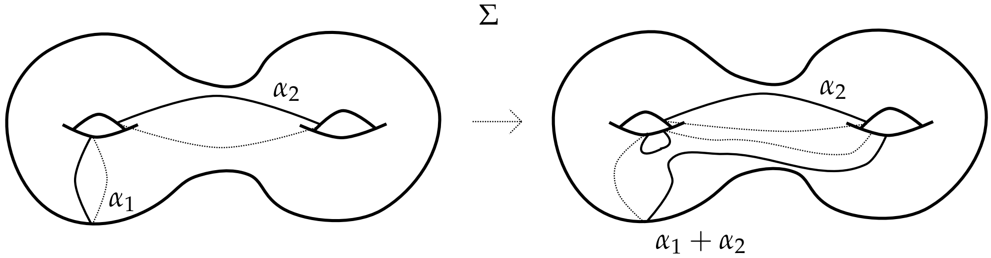


Figure 4.2: The curve $\alpha_1 + \alpha_2$ is the handleslide of α_1 over α_2.

metric $\mathbf{g}$ together with a choice of a gradient flowline, connecting the index zero and three critical points.

Then, there is an associated pointed Heegaard diagram H for M, whose surface Σ is the mid-level of the Morse function; the curves in α are the locus of points on Σ where the gradient flowline leaving the index one critical point meets Σ. Similarly, the curves in β are the one which flow from the index two critical point. Finally, w is the point on Σ that lies on the flowline.

If H is obtained in this manner, from a Morse function f, then we call f a Morse function *compatible* with H. Given a pointed Heegaard diagram for M, it is easy to construct a

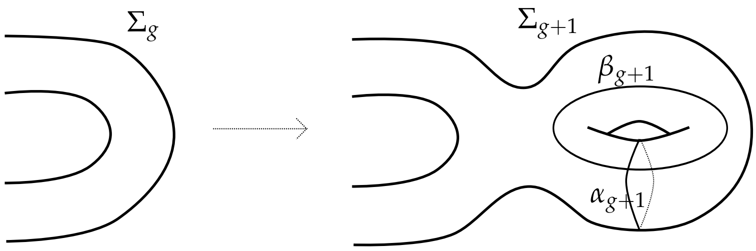


Figure 4.3: The new curves α_{g+1} and β_{g+1} are in standard position. This move corresponds to a connected sum with S^3.

compatible Morse function f. Moreover, we have the following result.

Theorem 4.1.1 (Ozsváth and Szabó) *Any two pointed Heegaard diagrams for M can be connected by a finite sequence of the following moves:*

1. *isotopies and handleslides of the curves in α (or β), supported in the complement of w, see Figure 4.2;*

2. *stabilizations (and their inverse), see Figure 4.3.*

Proof. The proof is sketched in [72]. The fact that every two (unpointed) Heegaard diagrams can be connected by such moves follows from standard Morse theory. This can be done in the complement of a single basepoint because we can trade an isotopy across the basepoint for a sequence of handleslides in the opposite direction [68]. $\square$

Using the basepoint we obtain additional information when the 3-manifold is equipped with a Spin^c structure. In this thesis, since we are working on 3-manifolds, the definition of *Spin^c structure* will be the one given by Turaev in [88]: an isotopy class, away from a point, of nowhere vanishing vector fields on the manifold.

We define the *g-th symmetric power* of a genus g surface Σ as the manifold

$$\mathrm{Sym}^g(\Sigma) = \frac{\overbrace{\Sigma \times ... \times \Sigma}^{g \text{ times}}}{(x_1, ..., x_g) = \left(x_{\sigma(1)}, ..., x_{\sigma(g)}\right)}$$

where $\sigma \in \mathcal{S}_g$, the group of permutations of g elements. Consider the g-dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g$$

in $\mathrm{Sym}^g(\Sigma)$.

Given a pointed Heegaard diagram $(\Sigma, \alpha, \beta, w)$ for the 3-manifold M, we have a well-defined map

$$\mathfrak{s}_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(M)$$

which sends an *intersection point* $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ into the Spin^c structure $\mathfrak{s}_w(x)$ on M, as described in [72]. The map is defined as follows. An intersection point x corresponds to a g-tuple of gradient flowlines which connect all the index one and two critical points. Let γ_x be the union of gradient flowlines passing through each $x_i \in x$, and γ_w be the union of gradient flowlines passing through each $w_i \in w$.

The closure of $\gamma_x \cup \gamma_w$ is a collection of arcs whose boundaries consist of all the critical points of f, which is a compatible Morse function. Moreover, each component contains a pair of critical points whose indices have opposite parities. Thus, we can modify the gradient vector field in an arbitrarily small neighborhood of $\gamma_x \cup \gamma_w$ to obtain a new vector field which vanishes nowhere in M. We call this vector field $\pi_w(x)$ and its homotopy class, in the sense of [88], $\mathfrak{s}_w(x)$.

The space $\mathrm{Spin}^c(M)$ is (not canonically) identified with $H^2(M; \mathbb{Z})$. For this reason we think of $\mathfrak{s}_w(x)$ as a 2-cohomology class in M.

4.1.2 Links

In the case of links, we need *(balanced) multi-pointed Heegaard diagrams*. These diagrams are defined as a quintuple $(\Sigma, \alpha, \beta, w, z)$, but the objects are slightly different. In fact, now the sets α and β consist of $g + n - 1$ curves, where $n \geqslant 1$ is an integer, and they are such that each one spans a g-dimensional subspace of $H_1(\Sigma; \mathbb{Z})$. Moreover, say $S_1, ..., S_n$ are the connected components of $\Sigma \setminus \alpha$ and $T_1, ..., T_n$ are the ones of $\Sigma \setminus \beta$, we have that w and z are two sets of n basepoints in Σ such that $w_i, z_i \in S_i \cap T_i$ for every $i = 1, ..., n$. See Figure 4.4.

A multi-pointed Heegaard diagram gives rise to a 3-manifold M together with an oriented link L in M as follows. Start from $[-1, 1] \times \Sigma$ and attach 3-dimensional 1-handles along copies of the α-curves in $\{-1\} \times \Sigma$; then attach three-dimensional 2-handles along the copies of the β-curves in $\{1\} \times \Sigma$. The result has $2n$ copies of S^2 as boundary, which we can close off with $2n$ 3-handles to get the closed 3-manifold M.

When we connect the basepoints in the components S_i and T_i, in the handlebodies given by the α-curves and β-curves, with unknotted and unlinked arcs, we get an embedded closed 1-manifold L in M. Next, we orient the link L so that the portions of the link

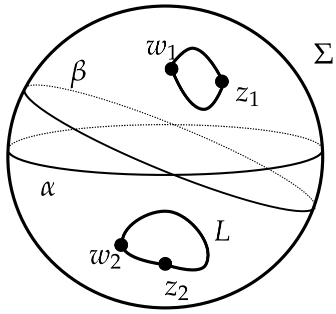


Figure 4.4: A very simple multi-pointed Heegaard diagram for the 2-component unlink in S^3.

in the α-handlebody point away from the w-basepoints and into the z-basepoints. The resulting link L has n components.

Indeed, any oriented link in any 3-manifold can be presented by an appropriate multi-pointed Heegaard diagram. Given an n-component link, one can find a self-indexing Morse function $f : M \to \mathbb{R}$ with n index zero and three critical points, and $g + n - 1$ index one and two critical points; with the additional property that there are two n-tuples of flowlines γ_w and γ_z, connecting all the index three and index zero critical points, so that our link L can be realized as the difference $\gamma_z - \gamma_w$. Such a Morse function gives rise to a multi-pointed Heegaard diagram for L in M.

Theorem 4.1.2 (Ozsváth and Szabó) *Every two multi-pointed Heegaard diagrams for the same n-component link L in M can be connected by a finite sequence of moves of the following types:*

1. *isotopies and handleslides of the curves in α (or β), supported in the complement of w and z;*
2. *stabilizations (and their inverse).*

Proof. The claim follows from Morse theory in the usual manner. □

In the previous subsection we saw that, starting from $(\Sigma, \alpha, \beta, w, z)$, we define two maps $\mathfrak{s}_w$ and $\mathfrak{s}_z$ from the set of the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to the one of Spin^c structures over M. The relation between these two maps is given in the following proposition.

Proposition 4.1.3 *Let $(\Sigma, \alpha, \beta, w, z)$ be a multi-pointed Heegaard diagram for a link L in a 3-manifold M. Then we have that*

$$\mathfrak{s}_w(x) - \mathfrak{s}_z(x) = PD[L]$$

for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Proof. It is proved in [68] that $\mathfrak{s}_w(x) - \mathfrak{s}_z(x)$ coincides with the Poincaré dual of $[\gamma]$, where γ is a curve in Σ such that $\alpha_i \cdot \gamma = 1$ for every $i = 1, ..., g$. The link L respects such conditions. □

As in Section 2.1, let us call M_L the 3-manifold with boundary $M \setminus \mathring{\nu}(L)$. When L has n components we have that ∂M_L consists of n disjoint tori. On this kind of 3-manifold we define a *relative* Spin^c *structure* as in [72]: the isotopy class, away from a point, of a nowhere vanishing vector field such that the restriction on each boundary torus coincides with the canonical one. This is the vector field induced by the Spin structure on the boundary tori of M_L; each structure is given by the product of the extendable Spin structure on the circle (see [72, 88]).

We denote the set of the relative Spin^c structures on M_L by $\text{Spin}^c(M, L)$; then we have an identification of $\text{Spin}^c(M, L)$ with the relative cohomology group $H^2(M_L, \partial\nu(L); \mathbb{Z})$. Moreover, from [72] we know that, given a multi-pointed Heegaard diagram, we construct a map

$$\mathfrak{s}_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \text{Spin}^c(M, L)$$

applying the following procedure.

Let γ be a gradient flowline connecting an index zero and index three critical point, and let $\nu(\gamma)$ denote a neighborhood of this flowline. One can construct a nowhere vanishing vector field v over $\nu(\gamma)$, which has an integral flowline P which enters $\nu(\gamma)$ from its boundary, contains γ as a subset and then exits $\nu(\gamma)$.

Let f be an orientation-preserving involution of $\nu(\gamma)$. We can arrange for $-v|_{\partial\nu(\gamma)}$ to agree with $f^*(v|_{\partial(\nu(\gamma)\cup P)})$. Indeed, we can construct v in the way that the difference $v - f^*(v)$ is the Poincaré dual of a meridian for γ, thought of as an element of $H_1(\nu(\gamma) \setminus P; \mathbb{Z})$.

Armed with this vector field, we can now define the map $\mathfrak{s}_{w,z}$. Fix a Morse function f, compatible with the Heegaard diagram. Given $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, consider the flowlines γ_x, γ_w and γ_z. We replace the gradient vector field in a neighborhood of γ_x so as not to vanish there. Similarly, we do the same in a neighborhood of γ_w, using v. In fact, arranging for P to consist of arcs on $\gamma_z \cup \gamma_w$, we obtain in this manner a vector field on M which contains L as a closed orbit. It is easy to see that this is equivalent to a vector field on M_L which is a standard non-vanishing vector field on the boundary tori.

Clearly the relative Spin^c structure $\mathfrak{s}_{w,z}(x)$ extends to the actual Spin^c structure $\mathfrak{s}_w(x)$. Furthermore, Poincaré duality gives that

$$H^2(M_L, \partial\nu(L); \mathbb{Z}) \cong H_1(M_L; \mathbb{Z}) \cong H_1(L; \mathbb{Z}) \oplus H_1(M; \mathbb{Z}) \cong \mathbb{Z}^n \oplus H^2(M; \mathbb{Z})$$

and if M is a rational homology sphere then $H^2(M; \mathbb{Z})$ is a finite group. A basis of the image of $H_1(L; \mathbb{Z}) \cong \mathbb{Z}^n$ in $H_1(M_L; \mathbb{Z})$ is given by the homology classes $\{[\mu_i]\}_{i=1,...,n}$; where μ_i is the meridian of the i-th component of L (Figure 1.1). Hence, the isomorphism between $\text{Spin}^c(M, L)$ and $H^2(M_L, \partial\nu(L); \mathbb{Z})$ is given by

$$\mathfrak{s}_{w,z} \longleftrightarrow c_1(\mathfrak{s}_{w,z}) - \sum_{i=1}^{n} \text{PD}[\mu_i] \,,$$

where $c_1(\cdot)$ denotes the first Chern class, see [60]. From now on, we think of $\mathfrak{s}_{w,z}$ as a relative 2-cohomology class under this identification. In particular, when M is a rational homology sphere we can write

$$\mathfrak{s}_{w,z}(x) = \sum_{i=1}^{n} c_i \cdot \text{PD}[\mu_i] + \mathfrak{s}_w(x) \quad \text{for any } x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \,,$$

where c_i is an integer for $i = 1, ..., n$.

Let us consider a multi-pointed Heegaard diagram $H = (\Sigma, \alpha, \beta, w, z)$, which represents an n-component link L in a rational homology sphere M. We easily obtain two diagrams for the link $-L$, that is gotten from L by reversing the orientation. The first diagram is $H_1 = (-\Sigma, \beta, \alpha, w, z)$, while the second one is $H_2 = (\Sigma, \alpha, \beta, z, w)$. Let us denote with $\mathfrak{s}'_{w,z}(x)$ the relative Spin^c structure induced by x in H_1 and with $\mathfrak{s}_{z,w}(x)$ the corresponding one in H_2. Then we can prove the following proposition.

Proposition 4.1.4 *Let $H = (\Sigma, \alpha, \beta, w, z)$ be a multi-pointed Heegaard diagram for L in M as above. Then we have that*

$$\mathfrak{s}'_{w,z}(x) = -\mathfrak{s}_{w,z}(x) \quad \text{and} \quad \mathfrak{s}_{z,w}(x) = \mathfrak{s}_{w,z}(x) - PD[L]$$

for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Proof. For the first statement, note that if f is a Morse function compatible with H then $-f$ is a Morse function compatible with H_1. It is now a straightforward consequence of its definition that if $\mathfrak{s}_{w,z}$ is represented by v then $-v$ represents $\mathfrak{s}_{w,z}$. Thus, the claim follows.

For the second observation, note that the vector fields v and w, representing the non-torsion parts of $\mathfrak{s}_{w,z}(x)$ and $\mathfrak{s}_{z,w}(x)$ respectively, differ only in a collar neighborhood of the boundary. In fact, one can see that in this neighborhood, they can be made isotopic away from a neighborhood of the meridian, where they point in opposite directions. in this way, we have proved that

$$\mathfrak{s}_{z,w}(x) - \mathfrak{s}_z(x) = \mathfrak{s}_{w,z}(x) - \mathfrak{s}_w(x)$$

and then we conclude by applying Proposition 4.1.3. □

We briefly recall that we can define an *unbalanced multi-pointed Heegaard diagram* as a multi-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ where we drop the condition that both w_i and z_i are in the intersection between the i-th connected component of $\Sigma \setminus \alpha$ and the one of $\Sigma \setminus \beta$; instead we only require that there is exactly one element from w and z inside each component of $\Sigma \setminus \alpha$ and $\Sigma \setminus \beta$. For this kind of diagrams we have that the resulting link has at most n components; the other results in this section still apply.

4.2 Open book decompositions

4.2.1 Adapted open book decompositions for links in 3-manifolds

Let us start this section by taking a 3-manifold M. We say that an *open book decomposition* for M is a pair (B, π) where

- the binding B is a smooth link in M;
- the map $\pi : M \setminus B \to S^1$ is a locally trivial fibration such that $\overline{\pi^{-1}(\theta)} = S_\theta$ is a compact surface with $\partial S_\theta = B$ for every $\theta \in S^1$. The surfaces S_θ are called *pages* of the open book.

Moreover, in the case that M comes equipped with a contact structure ξ, we say that the pair (B, π) *supports* ξ if we can find a 1-form α for ξ such that

- the 2-form $\mathrm{d}\alpha$ is a positive area form on each page S_θ;
- we have $\alpha > 0$ on the binding B.

Let us consider the page $S_1 = \overline{\pi^{-1}(1)}$; that is a connected, oriented, compact surface of genus g and with l boundary components. Suppose that an n-component Legendrian link L in (M, ξ) sits inside S_1 and its components represent n independent elements in $H_1(S_1; \mathbb{F})$, where we denote by $\mathbb{F}$ the field with two elements. We say that a collection of disjoint, simple arcs $A = \{a_1, ..., a_{2g+l+n-2}\}$ is a *system of generators for* S_1 *adapted* to L if the following conditions hold:

1. the subset $A^1 \sqcup A^2 \subset A$, where $A^1 = \{a_1, ..., a_n\}$ and $A^2 = \{a_{n+1}, ..., a_{2g+l-1}\}$, is a basis of $H_1(S_1, \partial S_1; \mathbb{F}) \cong \mathbb{F}^{2g+l-1}$;

2. we have that $L \pitchfork a_i = \{1 \text{ pt}\}$ for every $a_i \in A^1$ and $L \cap a_i = \emptyset$ for $i > n$. The arcs in the subset A^1 are called *distinguished arcs*;

3. the subset $A^3 = \{a_{2g+l}, ..., a_{2g+l+n-2}\}$ is such that the surface $S_1 \setminus (A^2 \sqcup A^3)$ has n connected components; each one containing exactly one component of L.

   The arcs in $A^2 \sqcup A^3$ that appear twice on the same component of the boundary of $S_1 \setminus (A^2 \sqcup A^3)$ are called *dead arcs*; the others *separating arcs*;

4. the elements in A^3 are separating arcs such that the disk $D = S_1 \setminus (A^1 \sqcup A^2)$ is disconnected into n disks and every arc in A^3 separates a unique pair of components of D.

An example of adapted system of generators is shown in Figure 4.5. With this definition in

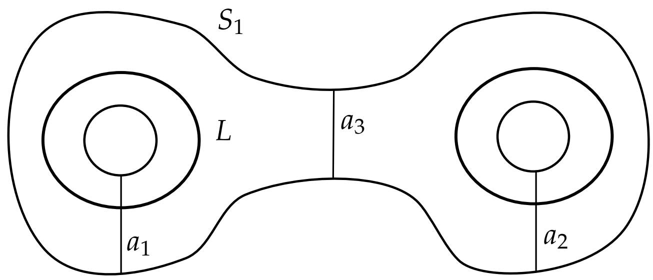


Figure 4.5: A system of generators adapted to the link L.

place, we say that the triple (B, π, A) is an *open book decomposition adapted* to the Legendrian link L if

- the pair (B, π) is compatible with M and supports ξ;
- the link L is contained in the page S_1;
- the n components of L represent independent elements in $H_1(S_1; \mathbb{Z})$;
- the set A is a system of generators for S_1 adapted to L.

In this case we also say that the adapted open book decomposition (B, π, A) is compatible with the triple (L, M, ξ). It is important to observe that, since the components of L are required to be independent in homology, we only consider open book decompositions with pages not diffeomorphic to a disk.

We can prove that open book decompositions adapted to Legendrian links always exist. In order to do this we recall the definition of contact cell decomposition (of a contact 3-manifold) and ribbon of a Legendrian graph. A *contact cell decomposition* of (M, ξ), see [27], is a finite CW-decomposition of M such that

1. the 1-skeleton is a connected Legendrian graph;
2. each 2-cell E satisfies $\mathrm{tb}(\partial E) = -1$;
3. the contact structure ξ is tight when restricted to each 3-cell.

Moreover, if we have a Legendrian link $L \hookrightarrow (M, \xi)$ then we also suppose that

4. the 1-skeleton contains L.

Denote the 1-skeleton of a contact cell decomposition of (M, ξ) with G. Then G is a Legendrian graph and its *ribbon* is a compact surface S_G satisfying:

- S_G retracts onto G;
- $T_pS_G = \xi_p$ for every $p \in G$;
- $T_pS_G \neq \xi_p$ for every $p \in S \setminus G$.

We say that an adapted open book decomposition (B, π, A), compatible with the triple (L, M, ξ), comes from a contact cell decomposition if $S = \overline{\pi^{-1}(1)}$ is a ribbon of the 1-skeleton of (M, ξ).

Theorem 4.2.1 *Every Legendrian link L in a contact 3-manifold (M, ξ) admits an adapted open book decomposition (B, π, A), which is compatible with the triple (L, M, ξ). Moreover, the contact framing of L coincides with the framing induced on L by the page S_1.*

Proof. In order to find an open book decomposition (B, π) which comes from a contact cell decomposition of (M, ξ), such that the link L is contained in S_1, we simply include the

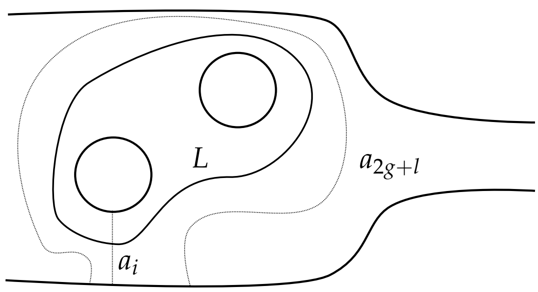


Figure 4.6: We add a new separating arc which is parallel to L_i except near the distinguished arc.

Legendrian link L in the 1-skeleton of the contact cell decomposition. Hence, it results that the page S_1 is precisely a ribbon of the 1-skeleton of (M, ξ). This argument also gives that the two framings of L agree.

The components of L are independent because it is easy to see from the construction that there is a collection of disjoint, properly embedded arcs $\{a_1, ..., a_n\}$ in S_1 such that

$$L_i \pitchfork a_i = \{1 \text{ pt}\} \quad \text{and} \quad L_i \cap \left(\bigcup_{j \neq i} a_j \right) = \emptyset$$

for every i. To conclude we only need to show that there exists a system of generators $A = \{a_1, ..., a_{2g+l+n-2}\}$ for S_1 which is adapted to L.

The arcs $a_1, ..., a_n$ are taken as before. If we complete L to a basis of $H_1(S;\mathbb{F})$ then Poincaré-Lefschetz duality gives a basis $\{a_1, ..., a_{2g+l-1}\}$ with the same property. We define $a_{2g+l}, ..., a_{2g+l+n-2}$ in the following way. Each new separating arc is parallel to L_i, extended by following the distinguished arc to the boundary of S_1 as in Figure 4.6. Clearly, it disconnects the surface, because the first $2g + l - 1$ arcs are already a basis of $H_1(S, \partial S; \mathbb{F})$. If one of the components contains no distinguished arcs, like in Figure 4.7, then we choose the other endpoint of a_i to extend the arc. □

In this thesis we use adapted open book decompositions to present Legendrian links in contact 3-manifolds. Moreover, we study how to relate two different open book decom-

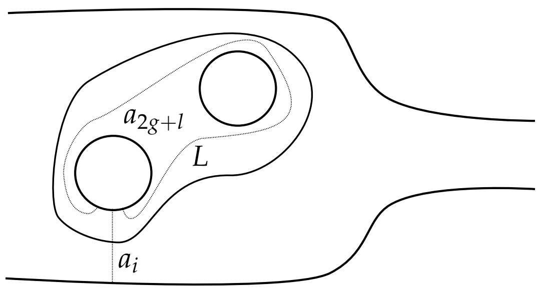


Figure 4.7: The picture appears similar to Figure 4.6, but this time the new arc follows the distinguished arc in the opposite direction.

positions representing isotopic Legendrian links; this is done in Chapter 6.

4.2.2 Abstract open books

An *abstract open book* is a quintuple $(S, \Phi, \mathcal{A}, z, w)$ defined as follows. We start with the pair (S, Φ). We have that $S = S_{g,l}$ is an oriented, connected, compact surface of genus g and with l boundary components, not diffeomorphic to a disk. While Φ is the isotopy class of a diffeomorphism of S into itself which is the identity on ∂S. The class Φ is called the *monodromy*.

Theorem 4.2.2 (Thurston and Winkelnkemper) *The pair (S, Φ) determines a contact 3-manifold.*

Proof. We begin by observing that, starting from (S, Φ), we get a 3-manifold M_Φ as follows:

$$M_\Phi = S_\Phi \cup_\psi \left(\coprod_l S^1 \times D^2 \right) ,$$

where l is the number of boundary components of S and S_Φ is the mapping torus of Φ. By this we mean

$$S \times [0,1] \big/ \sim ,$$

where $\sim$ is the equivalence relation $(\phi(x), 0) \sim (x, 1)$ for every $x \in S$. Finally, $\cup_\psi$ means that the diffeomorphism ψ is used to identify the boundaries of the two manifolds.

For each boundary component S_i of S, the map $\psi : \partial(S^1 \times D^2) \to S_i \times S^1 \subset S_\Phi$ is defined to be the unique, up to isotopy, diffeomorphism that takes $S^1 \times \{p\}$ to S_i, where $p \in \partial D^2$, and $\{q\} \times \partial D^2$ to

$$\{q'\} \times [0,1] \big/ \sim = S^1 ,$$

where $q \in S^1$ and $q' \in \partial S$.

The fact that (S, Φ) also determines a contact structure on M_Φ was proved in [87]. $\square$

The set $\mathcal{A}$ consists of two collections of properly embedded arcs, $B = \{b_1, ..., b_{2g+l+n-2}\}$ and $C = \{c_1, ..., c_{2g+l+n-2}\}$ in S with $n \geqslant 1$, such that all the arcs in B are disjoint, all the

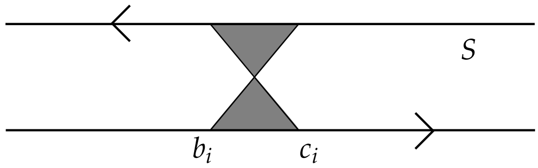


Figure 4.8: Two arcs in strip position.

arcs in C are disjoint and each pair b_i, c_i appears as in Figure 4.8. We suppose that each strip, the grey area between b_i and c_i, is disjoint from the others. We also want B and C to represent two systems of generators for the relative homology group $H_1(S, \partial S; \mathbb{F})$. In this way, if we name the strips $\mathcal{A}_i$, we have that $S \setminus \bigcup b_i$, $S \setminus \bigcup c_i$ and $S \setminus \bigcup \mathcal{A}_i$ have exactly n connected components.

Finally, z and w are two sets of basepoints: $w = \{w_1, ..., w_n\}$ and $z = \{z_1, ..., z_n\}$. We require these sets to have the following properties:

- there is a z_i in each component of $S \setminus \bigcup \mathcal{A}_i$, with the condition that every component contains exactly one element of z;

- each w_i is inside one of the strips $\mathcal{A}_i$, between the arcs b_i and c_i, with the property that every strip contains at most one element of w. See Figure 4.9;

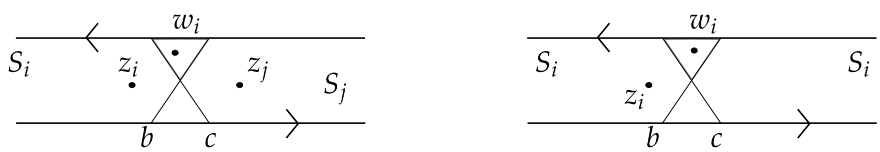


Figure 4.9: On the left S_i and S_j are different components of $S \setminus \bigcup \mathcal{A}_i$.

- we choose z and w in a way that each component of $S \setminus B$ and $S \setminus C$ contains exactly one element of z and one element of w.

We can draw an n-component link inside S using the following procedure: we go from the z's to the w's by crossing B and from w to z by crossing C, as shown in Figure 4.10. Moreover, we observe that the components of the link are independent in S.

Every abstract open book $(f(S), f \circ \Phi \circ f^{-1}, f(\mathcal{A}), f(z), f(w))$, obtained from $B = (S, \Phi, \mathcal{A}, z, w)$ by applying a diffeomorphism f of S into itself, is said to be *equivalent* to B. Then Theorem 4.2.2 gives the following corollary.

Corollary 4.2.3 *Two equivalent pairs (S, ϕ) and $(f(S), f \circ \Phi \circ f^{-1})$ determine contactomorphic contact 3-manifolds.*

Figure 4.10: The link is oriented accordingly to the basepoints.

An abstract open book B as above also determines a multi-pointed Heegaard diagram $(S \cup -S, b \cup \overline{b}, c \cup \overline{c}, z, w)$ and then a link in a 3-manifold up to (smooth) isotopy. Here the arcs $\overline{b}$ and $\overline{c}$ are defined as $h(b)$ and $(h \circ \Phi)(c)$ respectively, where $h : S \to -S$ is the Identity, seen as an unoriented diffeomorphism. Using this procedure we described we can prove the following theorem.

Theorem 4.2.4 (Legendrian realization theorem) *Every abstract open book $(S, \Phi, \mathcal{A}, z, w)$ determines a Legendrian link $\mathcal{L}$ in a contact 3-manifold (M, ξ). Furthermore, equivalent abstract open books give rise to contactomorphic links.*

Proof. The contact manifold (M, ξ) is obtained from the pair (S, Φ) as in Theorem 4.2.2. It is easy to check that its diffeomorphism type is the same of the manifold obtained from the Heegaard diagram $(S \cup -S, b \cup \overline{b}, c \cup \overline{c}, z, w)$. This gives us a smooth link $L \subset S \hookrightarrow M$.

The Legendrian realization $\mathcal{L}$ is obtained by applying the procedure described in [57] to every component of L. We only need to be careful that the new components are actually disjoint. $\square$

We are now interested in proving that an adapted open book decomposition (B, π, A) always determines an abstract open book.

Proposition 4.2.5 *We can associate to an adapted open book decomposition (B, π, A), compatible with the triple (L, M, ξ), an abstact open book $(S, \Phi, \mathcal{A}, z, w)$ up to isotopy.*

Proof. The surface S is obviously the page $\overline{\pi^{-1}(1)}$. Now consider the subsets of unit complex numbers $I_\pm \subset S^1 \subset \mathbb{C}$ with non-negative and non-positive imaginary part. Since they are contractible, we have that $\pi|_{\pi^{-1}(I_\pm)}$ is a trivial bundle. This gives two diffeomorphisms between the pages S_1 and S_{-1}. The monodromy Φ is precisely the isotopy class of the composition of first diffeomorphism with the inverse of the second.

At this point, we want to define the strips $\mathcal{A}$. Hence, we need the collections of arcs B and C: starting from the system of generators A, which is adapted to $L \subset S$, we take them to be both isotopic to A, in "strip position" like in Figure 4.8 and such that L does not cross the intersections of the arcs in B with the ones in C. We only have an ambiguity on, following the orientation of L, which is the first arc intersected by L. To solve this problem we have to follow the rule that we fixed in Figure 4.10.

Now we need to fix the basepoints. We put the z's on L; exactly one on each component of $L \setminus (L \cap \mathcal{A})$. The points in z on different components of L stay in different domains because of Condition 3 in the definition of adapted system of generators. Then $S \setminus \mathcal{A}$ has n connected components, since the components of L are independent, and each of these contains exactly one element of z. Since the z's are outside of the strips then we have that each component of $S \setminus B$ and $S \setminus C$ also contains only one z_i.

The w's are still put on L, but inside the strips containing the n distinguished arcs. The points $w_1, ..., w_n$ correspond to $\mathcal{A}_1, ..., \mathcal{A}_n$. $\square$

Chapter 5

Heegaard Floer homology

5.1 Link Floer homology

5.1.1 Heegaard diagrams and J-holomorphic curves

Link Floer homology is an invariant for knots and links in 3-manifolds, discovered in 2003 by Ozsváth and Szabó [69] and independently by Rasmussen [78], in his PhD thesis. It is constructed, here in the case of rational homology 3-spheres, by associating some chain complexes to a Heegaard diagram, with the help of Lagrangian Floer homology, a construction from symplectic geometry given by Andreas Floer in [32]. In this thesis we only discuss two of these complexes, that we define in this chapter.

Starting from a genus g multi-pointed Heegaard diagram $H = (\Sigma, \alpha, \beta, w, z)$ for an n-component link L in M, consider the $(g+n-1)$-fold symmetric product of Σ, that we defined in Subsection 4.1.1, and the $(g+n-1)$-dimensional tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, which are submanifolds of $\mathrm{Sym}^{g+n-1}(\Sigma)$.

The surface Σ can be equipped with a complex structure. Then $\mathrm{Sym}^{g+n-1}(\Sigma)$ inherits a complex structure from the direct product of Σ with itself. In particular, this means that we can fix an open set of almost-complex structures on $\mathrm{Sym}^{g+n-1}(\Sigma)$ compatible with an appropriate symplectic structure, which exists from results in [75].

Consider the set $\{J(s)\}$, where $s \in [0,1]$, a path of such almost-complex structures and define a *J-holomorphic strip*, connecting $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, as a continuous map

$$u : [0,1] \times \mathbb{R} \longrightarrow \mathrm{Sym}^{g+n-1}(\Sigma)$$

satisfying the following conditions:

1. u maps $\{0\} \times \mathbb{R}$ into $\mathbb{T}_\beta$ and $\{1\} \times \mathbb{R}$ into $\mathbb{T}_\alpha$;

2. the 1-parameter family of paths $u_t = u|_{[0,1]\times\{t\}}$ converges uniformly to the constant path at x (or y) when t goes to $-\infty$ (or $+\infty$);

3. u satisfies
$$\frac{\partial u}{\partial s} + J(s)_{u(s+it)} \frac{\partial u}{\partial t} = 0\,.$$

Note that the strip $[0,1] \times \mathbb{R}$ is conformally equivalent to the unit disk. This means that we can use this notion in place of the one of J-holomorphic disk. Details can be found in [67, 68, 69, 72].

We define the set $\pi_2(x,y)$, where $x,y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, as the set of $\mathbb{Z}$-linear combinations of the domains $\{\mathcal{D}_k\}$, which are the components of $\Sigma \setminus \alpha \cup \beta$, such that

$$a + b = c + d + \begin{pmatrix} 1 & \text{if } p \in x \\ 0 & \text{otherwise} \end{pmatrix} + \begin{pmatrix} -1 & \text{if } p \in y \\ 0 & \text{otherwise} \end{pmatrix} ;$$

where a, b, c, d are multiplicities of domains which appear like in Figure 5.1. For every $\phi \in \pi_2(x,y)$ we call

$$n_{z_i}(\phi) = \left|\phi \cap \{z_i\} \times \mathrm{Sym}^{g+n-2}(\Sigma)\right|$$

and

$$n_{w_i}(\phi) = \left|\phi \cap \{w_i\} \times \mathrm{Sym}^{g+n-2}(\Sigma)\right| ;$$

where here we mean algebraic intersection. Moreover, we say that

$$n_z(\phi) = \sum_{i=1}^{n} n_{z_i}(\phi) \qquad \text{and} \qquad n_w(\phi) = \sum_{i=1}^{n} n_{w_i}(\phi) .$$

We also define $\mathcal{M}(\phi)$ as the moduli space of holomorphic strips that connects x to y. The

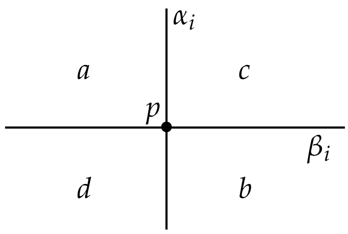


Figure 5.1: Corner point between four domains.

formal dimension of $\mathcal{M}(\phi)$, denoted by $\mu(\phi)$, is the *Maslov index*; moreover, we call $\widehat{\mathcal{M}}(\phi)$ the quotient $\mathcal{M}(\phi)/\mathbb{R}$ given by vertical translation of the strip.

For every Spin^c structure $\mathfrak{t}$ on M we define the group $cCFL^-(H,\mathfrak{t})$ as the free $\mathbb{F}[U]$-module over

$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}} = \left\{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \text{ such that } \mathfrak{s}_w(x) = \mathfrak{t}\right\} .$$

Moreover, we obtain another group by taking

$$\widehat{CFL}(H,\mathfrak{t}) = \frac{cCFL^-(H,\mathfrak{t})}{U=0} ;$$

this group is an $\mathbb{F}$-vector space, where $\mathbb{F}$ is again the field with two elements.

5.1.2 Gradings

We can put a bigrading on $cCFL^-(H,\mathfrak{t})$, where H is a Heeegaard diagram as in the previous subsection, if we keep the condition that the 3-manifold M is a rational homology 3-sphere.

For every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}}$ we denote with (W_x, J_x) an almost-complex 4-manifold, with M as boundary, such that the vector field $\pi(x)$, induced on M by J_x, is isotopic to the one given by x (and w), that we called $\pi_w(x)$ in Subsection 4.1.1. Such 4-manifolds always exist

[60], thus we define the *Maslov grading* as the absolute $\mathbb{Q}$-grading given by generalizing the Hopf invariant, defined in Subsection 3.1.2, as follows

$$M(x) = d_3\left(M, \pi_w(x)\right) = \frac{1}{4}\Big(c_1^2(W_x, J_x)[W_x, \partial W_x] - 3\sigma(W_x) - 2\chi(W_x) + 2\Big)$$

and we extend it to the whole $cCFL^-(H, \mathfrak{t})$ by saying that

$$M(Up) = M(p) - 2 \quad \text{where } p \in cCFL^-(H, \mathfrak{t}) \text{ is homogeneous}\,.$$

The Maslov grading is independent of the choice of (W_x, J_x).

Lemma 5.1.1 *If (W_x', J_x') is another almost-complex 4-manifold with the same properties of (W_x, J_x) then $d_3\left(M, \pi_w(x)\right)$ does not change.*

Proof. First, we have that $\mathfrak{t}$ is torsion, because M is a rational homology sphere. This implies that $c_1^2(W_x, J_x)[W_x, \partial W_x]$ is well-defined. Now we can find an almost-complex manifold (W, J) such that $\partial W = -M$ and the 2-plane field induced by J on $-M$ is isotopic to $\pi_w(x)$. This means that we can glue (W, J) to both our manifolds and obtain two closed almost-complex 4-manifolds. Since the evaluation on the square of the first Chern class, the signature and the Euler characteristic are all additive under gluing, the Hirzebruch's signature theorem gives that

$$\begin{aligned} c_1^2(W_x, J_x)[W_x, \partial W_x] - 3\sigma(W_x) - 2\chi(W_x) &= \\ -c_1^2(W, J)[W, \partial W] + 3\sigma(W) + 2\chi(W) &= \\ c_1^2(W_x', J_x')[W_x', \partial W_x'] - 3\sigma(W_x') - 2\chi(W_x') & \end{aligned}$$

and the claim follows. $\square$

The difference between the Maslov grading of two intersection points can be expressed easily.

Proposition 5.1.2 *Given $\phi \in \pi_2(x, y)$ we have that*

$$M(x) - M(y) = \mu(\phi) - 2n_w(\phi)$$

which is an integer.

Proof. From [72] we know that

$$\mu(\phi) - \mu(\psi) = 2(n_w(\phi) - n_w(\psi))$$

for every $\phi, \psi \in \pi_2(x, y)$. Since we can always find ψ such that $n_w(\psi) = 0$, we just apply the Atiyah-Singer index theorem and we obtain that

$$M(x) - M(y) = \mu(\psi) = \mu(\phi) - 2n_w(\phi)\,.$$

$\square$

The Maslov grading gives an $\mathbb{F}$-splitting

$$cCFL^-(H, \mathfrak{t}) = \bigoplus_{d \in \mathbb{Q}} cCFL_d^-(H, \mathfrak{t})\,;$$

moreover, the same holds for $\widehat{CFL}(H, \mathfrak{t})$.

The definition of $M(x)$ does not require that there is a link $L \to M$. The grading that we are going to define now, on the other hand, strictly requires that we take L, which has n components, into account. We call the *Alexander multi-grading* of $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}$ as the n-tuple $(s_1, ..., s_n)$ such that

$$\mathfrak{s}_{w,z}(x) = \sum_{i=1}^{n} 2s_i \cdot \mathrm{PD}[\mu_i] + \mathfrak{s}_w(x) \,.$$

Since L always admits a rational Seifert surface F, see Section 2.1, if t is the order of the link in M then we define the *Alexander absolute $\frac{\mathbb{Z}}{2t}$-grading* as follows:

$$A(x) = \sum_{i=1}^{n} s_i = \frac{\mathfrak{s}_{w,z}(x)[F]}{2t}$$

and we extend it on the whole $cCFL^-(H, \mathfrak{t})$ by saying that

$$A(Up) = A(p) - 1 \quad \text{where } p \in cCFL^-(H, \mathfrak{t}) \text{ is homogeneous} \,.$$

The Alexander grading is independent of the choice of the surface F.

Lemma 5.1.3 *If F' is another rational Seifert surface for L then the value of $A(x)$ does not change.*

Proof. The gluing $F \cup -F'$ defines a closed surface and then a 2-homology class in $M_L = M \setminus \mathring{\nu}(L)$ and M. Since $H_2(M; \mathbb{Z})$ is trivial, then $[F \cup -F']$ is null-homologous in it and also in $H_2(M_L; \mathbb{Z})$. This means that

$$0 = \mathfrak{s}_{w,z}(x)\left[F \cup -F'\right] = \mathfrak{s}_{w,z}(x)[F] - \mathfrak{s}_{w,z}(x)\left[F'\right] \,.$$

□

Like for the Maslov grading, we can compute the difference between the Alexander grading of two intersection points.

Proposition 5.1.4 *Given $\phi \in \pi_2(x, y)$ we have that*

$$A(x) - A(y) = n_z(\phi) - n_w(\phi)$$

which is again an integer.

Proof. From [72] we know that

$$\mathfrak{s}_{w,z}(x) - \mathfrak{s}_{w,z}(y) = 2 \cdot \sum_{i=1}^{n} \Big(n_{z_i}(\phi) - n_{w_i}(\phi) \Big) \cdot \mathrm{PD}[\mu_i] \,.$$

Then the claim follows by evaluating both sides of this equation with $[F]$, where F is a rational Seifert surface for L. □

We have another $\mathbb{F}$-splitting

$$cCFL^-(H, \mathfrak{t}) = \bigoplus_{d,s \in \mathbb{Q}} cCFL^-_{d,s}(H, \mathfrak{t})$$

and then again the same is true for $\widehat{CFL}(H, \mathfrak{t})$.

5.1.3 Differential

In the previous subsection we introduced the underlying space of our chain complexes; now we want to construct the differentials. We define ∂^- in the following way

$$\partial^- x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}}} \sum_{\substack{\phi \in \pi_2(x,y), \\ \mu(\phi)=1, n_z(\phi)=0}} \left| \widehat{\mathcal{M}}(\phi) \right| \cdot U^{n_w(\phi)} y$$

and

$$\partial^-(Ux) = U \cdot \partial^- x$$

for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}}$.

We note that, since we are interested in ϕ's that are image of some J-holomorphic disks, we have that $n_{z_i}(\phi), n_{w_i}(\phi) \geqslant 0$ [68]. This means that $n_z(\phi) = n_w(\phi) = 0$ if and only if $n_{z_i}(\phi) = n_{w_i}(\phi) = 0$ for every $i = 1, ..., n$.

The map ∂^- is well-defined if the diagram D is admissible, which means that every $\phi \in \pi_2(x,x)$ with $n_w(\phi) = 0$, representing a non-trivial homology class, has both positive and negative local multiplicities.

Lemma 5.1.5 *If M is a rational homology 3-sphere then, given a multi-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$, we can always achieve admissibility through isotopies.*

Proof. The proof is taken from [72]. We embed in Σ a tree Γ, whose n vertices are the basepoints in w. We claim that admissibility can be achieved by isotoping some of the β-curves in a regular neighborhood of Γ. More specifically, if γ is an arc connecting w_1 to w_2 in Γ then we perform an isotopy of the β's in a regular neighborhood of γ, in such a manner that there is a pair of arcs δ_1 and δ_2 so that $\delta_1 \cup \delta_2$ is isotopic to γ as an arc from w_1 to w_2, but δ_1 is disjoint from the β-circles, while δ_2 is disjoint from the α-circles.

Moreover, we find another pair of arcs δ_1' and δ_2' so that $\delta_1' \cup \delta_2'$ is isotopic to γ, only this time δ_1' is disjoint from the α-circles, while δ_2' is disjoint from the β-circles. Isotoping the β's in a regular neighborhood of all the edges in Γ as above, we obtain a Heegaard diagram that we claim is weakly admissible.

Suppose that $\phi \in \pi_2(x,x)$, which decomposes as $\phi = A + B$ where A has boundary on $\mathbb{T}_\alpha$ and B has boundary on $\mathbb{T}_\beta$, has the property that the oriented intersection number of ∂A with γ is non-zero. Then, at the intermediate endpoint of δ_1, we see that $A + B$ has local multiplicity given by $\partial A \cap \gamma$, while at the intermediate endpoint of δ_1' we have that $A + B$ has local multiplicity given by $\partial B \cap \gamma = -\partial A \cap \gamma$, since $\phi \in \pi_2(x,x)$ [68]. Thus, if for some edge in Γ it is $\partial A \cap \gamma \neq 0$, then $\phi = A + B$ has both positive and negative coefficients.

However, if $\phi = A + B$ as above and ∂A has algebraic intersection number equal to zero with each edge in Γ then, after subtracting off $n_{w_1}(A)$ of copies of Σ, we can write $\phi = A' + B'$, where A' and B' again have boundaries on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ and $n_{w_i}(A) = n_{w_i}(B) = 0$. Hence, a simple computation gives that $\phi = 0$. $\square$

We now prove that $\partial^- \circ \partial^- = 0$. See [72] for more details.

Theorem 5.1.6 (Ozsváth and Szabó) *We have that $\partial^-(\partial^- x)$ is zero for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}}$ and every Spin^c structure $\mathfrak{t}$ on M, which means that ∂^- is a differential.*

Proof. We consider the moduli space $\widehat{\mathcal{M}}(\phi)$, where $\phi \in \pi_2(x,z)$ is non-negative with $\mu(\phi) = 2$, x and z are intersection points and $n_z(\phi) = 0$. We use a structure theorem in [33, 69] which proves that there is a compact 1-dimensional manifold with boundary, whose interior is $\widehat{\mathcal{M}}(\phi)$, and its boundary is identified with

$$\bigcup_{y} \bigcup_{\substack{\phi_1 \in \pi_2(x,y), \phi_2 \in \pi_2(y,z) \\ \mu(\phi_1)=\mu(\phi_2)=1, \phi_1 * \phi_2 = \phi \\ n_z(\phi_1)=n_z(\phi_2)=0}} \widehat{\mathcal{M}}(\phi_1) \times \widehat{\mathcal{M}}(\phi_2) .$$

Since a compact 1-manifold has an even number of boundary points, we can conclude that

$$\sum_{y} \sum_{\substack{\phi_1 \in \pi_2(x,y), \phi_2 \in \pi_2(y,z) \\ \mu(\phi_1)=\mu(\phi_2)=1, \phi_1 * \phi_2 = \phi \\ n_z(\phi_1)=n_z(\phi_2)=0}} \left|\widehat{\mathcal{M}}(\phi_1)\right| \cdot \left|\widehat{\mathcal{M}}(\phi_2)\right| \cdot U^{n_w(\phi)} = 0 .$$

Adding up the left-hand-side over all $\phi \in \pi_2(x,z)$ with $\mu(\phi) = 2$ computes the z component of $\partial^-(\partial^- x)$, then $\partial^- \circ \partial^- = 0$. □

Using Lemma 5.1.5 and Theorem 5.1.6, we obtain the chain complexes

$$\left(cCFL^-(H,\mathfrak{t}), \partial^-\right) \quad \text{and} \quad \left(\widehat{CFL}(H,\mathfrak{t}), \widehat{\partial} = \partial^-|_{U=0}\right) .$$

Moreover, Proposition 5.1.2 leads to the following result.

Proposition 5.1.7 *For every intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}$ the element $\partial^- x$ is homogeneous with respect to the Maslov grading and $M(\partial^- x) = M(x) - 1$.*

Proof. If $U^{n_w(\phi)}y$, where $\phi \in \pi_2(x,y)$ and $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}$, is a monomial of $\partial^- x$ then we have that

$$M\left(U^{n_w(\phi)}y\right) = M(y) - 2n_w(\phi) = M(x) - \mu(\phi) = M(x) - 1 .$$

The last equality holds because in the differential we only consider ϕ's with Maslov index equal to 1. □

This implies that

$$\partial^-_d : cCFL^-_d(H,\mathfrak{t}) \longrightarrow cCFL^-_{d-1}(H,\mathfrak{t})$$

is the restriction of the differential to $cCFL^-_d(H,\mathfrak{t})$.

In the same way, Proposition 5.1.4 immediately gives a similar relation for the Alexander grading.

Proposition 5.1.8 *For every intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}$ the element $\partial^- x$ is homogeneous with respect to the Alexander grading and $A(\partial^- x) = A(x)$.*

Proof. If $U^{n_w(\phi)}y$, where $\phi \in \pi_2(x,y)$ and $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}$, is a monomial of $\partial^- x$ then we have that

$$A\left(U^{n_w(\phi)}y\right) = A(y) - n_w(\phi) = A(x) - n_z(\phi) = A(x) .$$

The last equality holds because in the differential we only consider ϕ's with $n_z(\phi) = 0$. □

Hence, the map

$$\partial^-_{d,s} : cCFL^-_{d,s}(H,\mathfrak{t}) \longrightarrow cCFL^-_{d-1,s}(H,\mathfrak{t})$$

is the restriction of the differential to $cCFL^-_{d,s}(H,\mathfrak{t})$.

5.1.4 Homology

In this section we define two versions of link Floer homology, corresponding to the chain complexes that we defined in the previous subsection. The first group is called *collapsed link Floer homology* and it is defined by

$$cHFL^{-}(H,\mathfrak{t}) = \bigoplus_{d,s\in\mathbb{Q}} cHFL^{-}_{d,s}(H,\mathfrak{t})\,,$$

where

$$cHFL^{-}_{d,s}(H,\mathfrak{t}) = \frac{\mathrm{Ker}\,\partial^{-}_{d,s}}{\mathrm{Im}\,\partial^{-}_{d+1,s}}\,.$$

The group $cHFL^{-}(H,\mathfrak{t})$ is a bigraded $\mathbb{F}$-vector space with a structure of finite dimensional $\mathbb{F}[U]$-module.

Theorem 5.1.9 (Ozsváth and Szabó) *The isomorphism type of the group $cHFL^{-}(H,\mathfrak{t})$ is invariant under smooth isotopy of the link $L \hookrightarrow M$. Which means that, if H' is another multi-pointed Heegaard diagram for a link isotopic to L in M, it is*

$$cHFL^{-}_{d,s}(H,\mathfrak{t}) \cong_{\mathbb{F}} cHFL^{-}_{d,s}(H',\mathfrak{t})$$

for every $d,s \in \mathbb{Q}$ and $\mathfrak{t} \in Spin^{c}(M)$.

Hence, we can denote $cHFL^{-}(H,\mathfrak{t})$ with $cHFL^{-}(M,L,\mathfrak{t})$. We leave the proof of this theorem for Section 5.3.

The second version that we want to study is the *hat link Floer homology group*

$$\widehat{HFL}(H,\mathfrak{t}) = \bigoplus_{d,s\in\mathbb{Q}} \widehat{HFL}_{d,s}(H,\mathfrak{t})\,;$$

given by

$$\widehat{HFL}_{d,s}(H,\mathfrak{t}) = \frac{\mathrm{Ker}\,\widehat{\partial}_{d,s}}{\mathrm{Im}\,\widehat{\partial}_{d+1,s}}\,.$$

The group $\widehat{HFL}(H,\mathfrak{t})$ is a finite dimensional, bigraded $\mathbb{F}$-vector space. Moreover, as the collapsed minus version, it is also a smooth isotopy invariant of links in 3-manifolds. In fact, we have an analogous theorem in this case.

Theorem 5.1.10 (Ozsváth and Szabó) *The isomorphism type of the group $\widehat{HFL}(H,\mathfrak{t})$ is invariant under smooth isotopy of the link $L \hookrightarrow M$. Which means that, if H' is another multi-pointed Heegaard diagram for a link isotopic to L in M, it is*

$$\widehat{HFL}_{d,s}(H,\mathfrak{t}) \cong_{\mathbb{F}} \widehat{HFL}_{d,s}(H',\mathfrak{t})$$

for every $d,s \in \mathbb{Q}$ and $\mathfrak{t} \in Spin^{c}(M)$.

Hence, we can denote $\widehat{HFL}(H,\mathfrak{t})$ with $\widehat{HFL}(M,L,\mathfrak{t})$. Theorems 5.1.9 and 5.1.10 have been proved in [69] for knots and generalized to links in [72].

The group $\widehat{HFL}$ has the following interesting symmetry property.

Proposition 5.1.11 *Given a smooth link L in a rational homology 3-sphere M, we have that*

$$\widehat{HFL}_{d,s}\left(M,L,\mathfrak{t}\right) \cong_{\mathbb{F}} \widehat{HFL}_{d-2s,-s}\left(M,L,-\mathfrak{t}+PD[L]\right)$$

for every $d,s \in \mathbb{Q}$ and $\mathfrak{t} \in Spin^c\left(M\right)$.

Proof. It follows from the relation between the two relative Spin^c structures obtained by reversing the orientation of L, see Subsection 4.1.2 before. More specifically, given $(\Sigma,\alpha,\beta,w,z)$ a multi-pointed Heegaard diagram for L in M, we call $H_1 = (\Sigma,\beta,\alpha,w,z)$ the first diagram for $-L$. Then Proposition 4.1.4 gives us the isomorphism

$$\Phi_1 : \widehat{HFL}_{d,s}\left(M,L,\mathfrak{t}\right) \longrightarrow \widehat{HFL}_{d,s}\left(M,-L,-\mathfrak{t}\right) .$$

In the same way, we call $H_2 = (\Sigma,\alpha,\beta,z,w)$, which also represents the link $-L$. Now from Propositions 4.1.3 and 4.1.4 we obtain the map

$$\Phi_2 : \widehat{HFL}_{d,s}\left(M,-L,-\mathfrak{t}\right) \longrightarrow \widehat{HFL}_{d-2s,-s}\left(M,L,-\mathfrak{t}+\mathrm{PD}[L]\right)$$

and the proof is completed by taking the composition of Φ_1 and Φ_2. □

We can use this property to normalize the relative grading $a(x,y) = n_z(\phi) - n_w(\phi)$, where $\phi \in \pi_2(x,y)$, given in Proposition 5.1.4. Hence, this can be seen as an equivalent definition of the Alexander grading; in fact, we have that $A(x)$ is the only normalization of $a(x,y)$ which respects the symmetry in Proposition 5.1.11. This property proves the following corollary.

Corollary 5.1.12 *For every multi-pointed Heegaard diagram $(\Sigma,\alpha,\beta,w,z)$, and $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have that*

$$2A(x) = M(x) - M_z(x)\,,$$

where $A(x)$ and $M(x)$ are the Alexander and Maslov grading of x and $M_z(x) = d_3\left(M,\pi_z(x)\right)$.

Proof. From Propositions 5.1.2 and 5.1.4 we have that the relative grading

$$b(x,y) = (M(x) - M(y)) - (M_z(x) - M_z(y))$$

coincides with $2a(x,y) = 2(A(x) - A(y))$. Therefore, we just need to show that $b(x,y)$ satisfies the symmetries in Proposition 5.1.11 and this can be done easily. □

5.2 Heegaard Floer homology of 3-manifolds

Let us take a multi-pointed Heegaard diagram $H = (\Sigma,\alpha,\beta,w)$, where the set w contains n basepoints. We can change the differential $\widehat{\partial}$ to $\widehat{\partial}_w$ as follows:

$$\widehat{\partial}_w x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}} \sum_{\substack{\phi \in \pi_2(x,y),\\ \mu(\phi)=1, n_w(\phi)=0}} \left|\widehat{\mathcal{M}}(\phi)\right| \cdot y$$

for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_\mathfrak{t}$ and $\widehat{\partial}_w$ is U-equivariant. The new complex $\left(\widehat{CF}(H,\mathfrak{t}),\widehat{\partial}_w\right)$ is well-defined. Moreover, its homology

$$\widehat{HF}(H,\mathfrak{t}) = \bigoplus_{d \in \mathbb{Q}} \widehat{HF}_d(H,\mathfrak{t})$$

is an invariant of the manifold, once we fix the number of basepoints in H, as stated in the following theorem.

Theorem 5.2.1 (Ozsváth and Szabó) *The isomorphism type of the group $\widehat{HF}(H,\mathfrak{t})$ is invariant under diffeomorphism of the Spin^c 3-manifold $(M,\mathfrak{t})$. In other words, if H' is another Heegaard diagram for a manifold diffeomorphic to $(M,\mathfrak{t})$, and with the same number of basepoints in H, then it is*

$$\widehat{HF}_d(H,\mathfrak{t}) \cong_{\mathbb{F}} \widehat{HF}_d(H',\mathfrak{t})$$

for every $d \in \mathbb{Q}$ and $\mathfrak{t} \in Spin^c(M)$.

When the cardinality of w is one we denote $\widehat{HF}(H,\mathfrak{t})$ with $\widehat{HF}(M,\mathfrak{t})$. The proof of this theorem follows the same techniques of the ones of Theorems 5.1.9 and 5.1.10, see [68].

It is easy to observe that if $H^{w,z} = (\Sigma, \alpha, \beta, w, z)$ is a multi-pointed Heegaard diagram, where each z_i and w_i are in the same component of $\Sigma \setminus \alpha \cup \beta$, then the differential $\widehat{\partial}$ coincides with $\widehat{\partial}_w$, which is defined from the diagram H, obtained from $H^{w,z}$ by dropping the basepoints z. Hence, we have the following proposition.

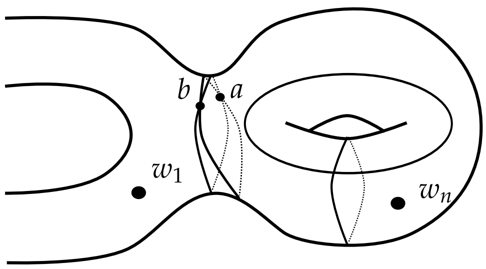


Figure 5.2: This stabilization represents a disjoint union of a link in a 3-manifold with an unknot.

Lemma 5.2.2 *Given a rational homology 3-sphere $(M,\mathfrak{t})$, we have that*

$$\widehat{HFL}(M, \bigcirc_n, \mathfrak{t}) \cong \widehat{HF}(M,\mathfrak{t}) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)}$$

and the isomorphism preserves the Maslov grading. Here, the symbol $\bigcirc_n$ denotes the n component unlink in M.

Proof. We reason by induction on n. If $n = 1$ then the claim follows immediately from the previous observation. For the general case, we use Theorem 5.2.1 and we obtain that, after some handleslides and stabilizations, a multi-pointed Heegaard diagram for $\bigcirc_n$ in M appears as in Figure 5.2. Then, using the inductive hypothesis, we obtain that we just have to prove that changing a Heegaard diagram H into H' by this type of stabilization corresponds to tensor product with $\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}$ in the homology group.

To do this we observe that an explicit computation gives that $M(x_a) = M(x)$ and $M(x_b) = M(x) - 1$, where x_a (or x_b) is given by adding the point a (or b) to the intersection point x; see Figure 5.2. Since every intersection point in the new complex needs to have one between a or b as a coordinate, we have an isomorphism of chain complexes

$$\widehat{CFL}(H',\mathfrak{t}) \longrightarrow \widehat{CFL}(H,\mathfrak{t}) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right).$$

Therefore, we conclude by noting that the differential is zero on $\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}$ and then the isomorphism descends in homology. $\square$

This lemma implies the following theorem.

Theorem 5.2.3 (Ozsváth and Szabó) *If the set w in the Heegaard diagram H contains n basepoints then we have that*

$$\widehat{HF}(H,\mathfrak{t}) \cong_{\mathbb{F}} \widehat{HF}(M,\mathfrak{t}) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)} . \tag{5.1}$$

More explicitly, it is

$$\widehat{HF}_d(H,\mathfrak{t}) \cong_{\mathbb{F}} \bigoplus_{i=0}^{n-1} \binom{n-1}{i} \widehat{HF}_{d+i}(M,\mathfrak{t})$$

for every $d \in \mathbb{Q}$ and $\mathfrak{t} \in Spin^c(M)$.

From Propositions 5.1.4 and 5.1.11 we see that the Alexander grading is always zero in the case of unlinks. Hence, we can express the hat link Floer homology of $\bigcirc_n$ entirely in terms of the homology of M:

$$\widehat{HFL}_{d,s}(M,\bigcirc_n,\mathfrak{t}) \cong \begin{cases} \bigoplus_{i=0}^{n-1} \binom{n-1}{i} \widehat{HF}_{d+i}(M,\mathfrak{t}) & \text{if } s = 0 \\ \{0\} & \text{otherwise} \end{cases} .$$

A particular case is given by the 2-component unlink $\bigcirc_2$ in S^3. In fact, we have that $\widehat{HFL}(\bigcirc_2) \cong \mathbb{F}_{(-1,0)} \oplus \mathbb{F}_{(0,0)}$; note that we do not specify the manifold when we work in the 3-sphere.

We conclude this section by computing the homology of the unlink also in the collapsed version. We observe that, for the same argument used to prove Lemma 5.2.2, the differential ∂^- coincides with $\widehat{\partial}$ too if we choose an appropriate diagram. Moreover, the generators of $cCFL^-(H,\mathfrak{t})$ and $\widehat{CF}(H,\mathfrak{t})$ are the same; with the difference that the first space is an $\mathbb{F}[U]$-module. This gives the following result.

Corollary 5.2.4 *The homology group $cHFL^-(M,\bigcirc_n,\mathfrak{t})$, seen as an $\mathbb{F}[U]$-module, is the tensor product of $\widehat{HFL}(M,\bigcirc_n,\mathfrak{t})$ with (the module) $\mathbb{F}[U]$ over $\mathbb{F}$.*

Using Lemma 5.2.2 we can restate this corollary by saying that

$$cHFL^-_{d,s}(M,\bigcirc_n,\mathfrak{t}) \cong \begin{cases} \bigoplus_{i=0}^{n-1} \binom{n-1}{i} \widehat{HF}_{d-2s+i}(M,\mathfrak{t}) & \text{if } s \leqslant 0 \\ \{0\} & \text{otherwise} \end{cases} .$$

5.3 Invariance

5.3.1 Choice of the almost-complex structures

In this section we give ideas for the proofs that link Floer homology is invariant under the moves in Section 4.1, that relates two multi-pointed Heegaard diagrams representing the same link L in the Spin^c 3-manifold $(M,\mathfrak{t})$, and the analytic input of the path of almost-complex structures $J(s)$ on the symmetric power and the complex structure on Σ, see Subsection 5.1.1. We always suppose that M is a rational homology 3-sphere.

Suppose that $H = (\Sigma, \alpha, \beta, w, z)$ is a multi-pointed Heegaard diagram for an n-component link in M. First, we argue that if we fix a complex structure on Σ then the link Floer homology groups are independent of the choice of $J(s)$ on $\mathrm{Sym}^{g+n-1}(\Sigma)$. Take two paths $J_0(s)$ and $J_1(s)$ on $\mathrm{Sym}^{g+n-1}(\Sigma)$; since the symmetric power is simply-connected, we can connect $J_0(s)$ and $J_1(s)$ with a 2-parameter family $J : [0,1] \times [0,1] \to \mathrm{Sym}^{g+n-1}(\Sigma)$, thought of as a 1-parameter family of paths indexed by $t \in [0,1]$ where $J_t(s)$ is the path obtained by fixing t. In fact, we can arrange that $J_t(s)$ is independent of t for t's near to zero or one, so that $J_t(s)$ can be naturally extended to the whole $\mathbb{R}$.

Then, as it is familiar in Floer theory, we can define an associated chain map

$$\Phi^-_{J_t(s)} : \left(cCFL^-(H,\mathfrak{t}), \partial^-_{J_0(s)}\right) \longrightarrow \left(cCFL^-(H,\mathfrak{t}), \partial^-_{J_1(s)}\right)$$

by

$$\Phi^-_{J_t(s)} x = \sum_y \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1, n_z(\phi)=0}} \left|\widehat{\mathcal{M}}_{J_t(s)}(\phi)\right| \cdot U^{n_w(\phi)} y \,,$$

where $\widehat{\mathcal{M}}_{J_t(s)}(\phi)$ denotes the moduli space of holomorphic strips with a time-dependent complex structure on the target.

The usual arguments from Floer theory then apply to show that $\Phi^-_{J_t(s)}$ is a chain map which induces an isomorphism in homology. We give a sketch of the proof of this fact. The details can be found in [68].

A transversality theorem in [68] shows that, for a generic path $J_t(s)$, the zero-dimensional components of the moduli spaces $\widehat{\mathcal{M}}_{J_t(s)}(\phi)$ are smoothly cut out and compact. Thus, the map $\Phi^-_{J_t(s)}$ is well-defined. To show that it is a chain map, we consider the ends of the 1-dimensional moduli spaces $\widehat{\mathcal{M}}_{J_t(s)}(\psi)$, where $\mu(\psi) = 2$. Then the same argument in the proof of Theorem 5.1.6 and Gromov's compactness [42] give that $\Phi^-_{J_t(s)}$ is also a chain map.

The fact that $\Phi^-_{J_t(s)}$ induces an isomorphism in homology is done by showing that the composition with $\Phi^-_{J_{1-t}(s)}$ is chain homotopic to the identity; we recall that we said what is a chain homotopy in Subsection 1.4.2. The chain homotopy is constructed by using a homotopy $J_{t,\tau}(s)$ between two 2-parameter families of complex structures. We can define a moduli space

$$\widehat{\mathcal{M}}_{J_{t,\tau}(s)}(\phi) = \bigcup_{\eta \in [0,1]} \widehat{\mathcal{M}}_{J_{t,\eta}(s)}$$

for each $\phi \in \pi_2(x,y)$. For a generic $J_{t,\tau}(s)$ this manifold has dimension $\mu(\phi)+1$. We define a map

$$H^-_{J_{t,\tau}(s)} x = \sum_y \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=0, n_z(\phi)=0}} \left|\widehat{\mathcal{M}}_{J_{t,\tau}(s)}(\phi)\right| \cdot U^{n_w(\phi)} y \,.$$

To see then that $H^-_{J_{t,\tau}(s)}$ is the chain homotopy between $\Phi^-_{J_t(s)} \circ \Phi^-_{J_{1-t}(s)}$ and the identity map, we reason in the same way as before [68].

Next, we see that the chain complex is independent of the complex structure $\mathfrak{J}$ on the surface Σ. To this end, we observe that the chain complexes remain unchanged under small perturbations of the path of almost-complex structures $J(s)$. Furthermore, we can approximate $J(s)$ with a symmetric path $J'(s)$, induced by a complex structure $\mathfrak{J}'$ close to $\mathfrak{J}$. This shows that the Floer homology is also independent of the choice of $\mathfrak{J}$, since the space of allowed complex structures over Σ is connected, because it is obtained from the space of all complex structures by removing a codimension two subset, see [68].

5.3.2 Isotopies

Consider two multi-pointed Heegaard diagram $H = (\Sigma, \alpha, \beta, w, z)$ and $H' = (\Sigma, \alpha', \beta', w, z)$ which differ by an isotopy. First of all, isotopies which preserve the condition that the α are transverse to the β can be thought of as variations in the metric, or equivalently the complex structure on Σ. Hence, the invariance of the Floer homology under such isotopies follows from Subsection 5.3.1.

It suffices then to show that the homology remains unchanged when a pair of canceling intersection points between, α_1 and β_1 is introduced. See Figure 5.3 for an illustration. Such an isotopy can be realized by moving α by an exact Hamiltonian diffeomorphism of Σ. We recall that, on a symplectic manifold, a 1-parameter family of real-valued functions H_t naturally gives rise to a unique 1-parameter family of Hamiltonian vector fields X_t, specified by

$$\omega(X_t, \cdot) = \mathrm{d}H_t \, ,$$

where the left-hand side denotes the contraction of the symplectic form ω with the vector field. A 1-parameter family of diffeomorphisms Ψ_t is said to be an *exact Hamiltonian isotopy* if it is obtained by integrating a Hamiltonian vector field, which means when

- Ψ_0 is the Identity;
- $\dfrac{\mathrm{d}\Psi_t}{\mathrm{d}t} = X_t$.

By taking a positive bump function h, supported in a neighborhood of a point which lies on α_1, and letting $f : \mathbb{R} \to [0,1]$ be a non-negative smooth function whose support is $(0,1)$, we can consider the Hamiltonian $H_t = f(t)h$. The corresponding diffeomorphism moves the curve α_1 slightly, without moving any of the other curves in α.

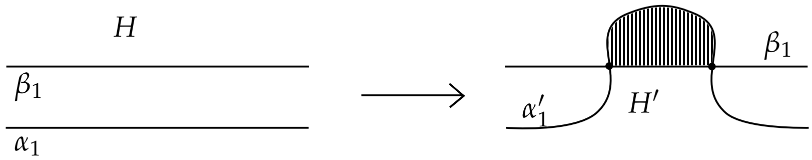


Figure 5.3: By moving the curve α_1 to α'_1 through a Hamiltonian isotopy, we introduce a pair of canceling intersection points.

We assume that the exact Hamiltonian is supported on $\Sigma \setminus (\alpha_2 \cup ... \cup \alpha_{g+n-1} \cup w \cup z)$ and that the Hamiltonian is supported in $[0,1]$. The isotopy Ψ_t induces an isotopy of $\mathbb{T}_\alpha$. We need to show that this isotopy gives a map on Floer homology, by imitating the usual constructions from Lagrangian Floer theory.

The map is induced by counting points in the zero-dimensional components of the moduli spaces with a time-dependent constraint. To be precise, here, we fix a 1-parameter family $J(s)$ of almost-complex structures on the symmetric power of Σ. Now, we have homotopy classes of holomorphic strips $\pi_2^{\Psi_t}(x,y)$, where x is an intersection point and $y \in \Psi_1(\mathbb{T}_\alpha) \cap \mathbb{T}_\beta$, which denote homotopy classes of maps satisfying the condition in Subsection 5.1.1 and such that

$$u(1+it) \in \Psi_1(\mathbb{T}_\alpha) \quad \text{and} \quad u(0+it) \in \mathbb{T}_\beta$$

for every $t \in [0,1]$.

Now, we define the map associated to the isotopy

$$\Gamma^-_{\Psi_t} : cCFL^-(H) \longrightarrow cCFL^-(H')$$

by the formula

$$\Gamma^-_{\Psi_t} x = \sum_y \sum_{\substack{\phi \in \pi_2^{\Psi_t}(x,y) \\ \mu(\phi)=1, n_z(\phi)=0}} \left| \widehat{\mathcal{M}}^{\Psi_t}(\phi) \right| \cdot U^{n_w(\phi)} y \, .$$

The important observation is that the moduli spaces considered have Gromov compactifications. This is shown in [68]. Then, the fact that $\Gamma^-_{\Psi_t}$ induces an isomorphism in homology can be proved in the same way as in Subsection 5.3.1.

5.3.3 Handleslides

Consider a multi-pointed Heegaard diagram $H' = (\Sigma, \alpha, \gamma, w, z)$ which is obtained from $(\Sigma, \alpha, \beta, w, z)$ by a single handleslide on a curve in β. Then there is a chain map

$$\Psi_{\alpha\beta\gamma} : cCFL^-(H, \mathfrak{t}) \longrightarrow cCFL^-(H, \mathfrak{t}')$$

defined by counting holomorphic triangles as follows

$$\Psi_{\alpha\beta\gamma} x = \sum_y \sum_{\substack{\phi \in \pi_2(x, \Theta_{\beta\gamma}, y) \\ \mu(\phi)=1, n_z(\phi)=0}} \left| \widehat{\mathcal{M}}(\phi) \right| \cdot U^{n_w(\phi)} y \, ,$$

where $\Theta_{\beta\gamma}$ is the canonical top-generator of the homology of the chain complex $\big(CF^-(\Sigma, \beta, \gamma, w), \partial^-_w\big)$, whose definition is in [72] together with all the details.

Let us denote with $\Psi^*_{\alpha\beta\gamma}$ the induced map in homology, then we note that

$$\mathfrak{s}_{w,z}(x) = \mathfrak{s}'_{w,z}\left(\Psi^*_{\alpha\beta\gamma}(x)\right) \, ,$$

where here $\mathfrak{s}'_{w,z}$ is the map associated to the Heegaard diagram H', since the triangles used in the definition of Ψ^* are disjoint from w and z.

Now, associativity of the triangle maps [72] gives that the map $(\Psi_{\alpha\gamma\beta'} \circ \Psi_{\alpha\beta\gamma})^*$ is an isomorphism, where

$$\Psi_{\alpha\gamma\beta'}(\Psi_{\alpha\beta\gamma}(x)) = \sum_y \sum_{\substack{\phi \in \pi_2(x, \Theta_{\beta\beta'}, y) \\ \mu(\phi)=1, n_z(\phi)=0}} \left| \widehat{\mathcal{M}}(\phi) \right| \cdot U^{n_w(\phi)} y$$

and β' is a small perturbation of β such that each β'_i intersects β_i in precisely two intersection points. Therefore, we have that the map $\Psi^*_{\alpha\beta\gamma}$ is also an isomorphism.

5.3.4 Stabilizations

Finally, suppose that $H' = (\Sigma', \alpha', \beta', w', z')$ is obtained from $H = (\Sigma, \alpha, \beta, w, z)$ by a stabilization. Then we immediately observe that, since we perform the stabilization away from w and z, the assignment to relative Spin^c structures is unaffected. Denote with c the intersection point that appears in the stabilization. Then it is easy to see that the intersection points in H' correspond to $\mathbb{T}'_\alpha \cap \mathbb{T}'_\beta = \mathbb{T}_\alpha \cap \mathbb{T}_\beta \times \{c\}$.

Now fix basepoint w_i in the same path-component of $\Sigma \setminus (\alpha \cup \beta)$ as $p \in \Sigma$, the point that we use to perform the connected sum that leads to Σ'. Let w'_i be the corresponding basepoint in Σ'. If $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $x' \in \mathbb{T}'_\alpha \cap \mathbb{T}'_\beta$ is the new intersection point $x \times \{c\}$ then the induced Spin^c structures $\mathfrak{s}_w$ and $\mathfrak{s}_{w'}$ agree, since the corresponding vector fields agree away from the 3-ball where the stabilization occurs.

Consider x and y intersection points and take $\phi \in \pi_2(x, y)$ such that $n_z(\phi) = 0$. Moreover, consider $x' = x \times \{c\}$ and $y' = y \times \{c\}$ and take $\phi' \in \pi_2(x', y')$ such that $n_{z'}(\phi') = 0$. Hence, using a result in [68] we have that, for certain special paths of almost-complex structures, the moduli space $\mathcal{M}(\phi')$ is identified with $\mathcal{M}(\phi) \times \{c\}$; together with its deformation theory, including the determinant line bundles. This implies that the chain complexes are identical and then the homology groups also coincide.

5.4 Multiplication by $U + 1$ in the link Floer homology group

Take a multi-pointed Heegaard diagram $H = (\Sigma, \alpha, \beta, w, z)$ for an n-component link L in the Spin^c 3-manifold $(M, \mathfrak{t})$, where M is a rational homology 3-sphere. Consider also the diagram $H^z = (\Sigma, \alpha, \beta, z)$. We have the following surjective $\mathbb{F}$-linear map:

$$F : cCFL^-(H, \mathfrak{t} + \mathrm{PD}[L]) \xrightarrow{U=1} \widehat{CF}(H^z, \mathfrak{t})\,;$$

which is given by setting U equal to 1.

The map F clearly commutes with the differentials; it is well-defined and surjective, because every intersection point x is such that $F(x) = x$. Moreover, if $s_z(x) = \mathfrak{t}$ then $s_w(x) = \mathfrak{t} + \mathrm{PD}[L]$ from Proposition 4.1.3.

Lemma 5.4.1 *The map F sends an element with bigrading (d, s) into an element with Maslov grading $d - 2s$.*

Proof. We have that

$$\begin{aligned} M_z(F(U^k x)) &= M_z(x) = M(x) - 2A(x) = \\ &= (M(x) - 2k) - 2(A(x) - k) = M(U^k x) - 2A(U^k x)\,, \end{aligned}$$

where the second equality follows from Corollary 5.1.12. □

The map F induces a map F_* in homology:

$$F_* : cHFL^-\left(M, L, \mathfrak{t} + \mathrm{PD}[L]\right) \xrightarrow{U=1} \widehat{HF}\left(M, \mathfrak{t}\right) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)}\,.$$

We now recall that, since the link Floer homology group is an $\mathbb{F}[U]$-module and $\mathbb{F}$ is a field, it is

$$cHFL^-\left(M, L, \mathfrak{t}'\right) \cong \mathbb{F}[U]^r \oplus T\,,$$

where r is an integer and T is the torsion $\mathbb{F}[U]$-module; see Subsection 1.5.2.

Lemma 5.4.2 *The following two statements hold:*

1. *we have that $F(x) = 0$ and x is homogeneous with respect to the Alexander grading if and only if $x = 0$;*

2. *we have that $F_*[x] = [0]$ and $[x_i]$ is homogeneous with respect to the Alexander grading, for every $i = 1, ..., r$, if and only if $[x]$ is torsion. Here the $[x_i]$'s are a decomposition of $[x]$ in the torsion-free quotient of $cHFL^-(M, L, \mathfrak{t} + \mathrm{PD}[L])$.*

Proof. 1. The if implication is trivial. For the only if, suppose that $F(x) = 0$; this gives that

$$x = (1+U)\lambda_1(U)y_1 + ... + (1+U)\lambda_t(U)y_t \,,$$

where $\lambda_i(U) \in \mathbb{F}[U]$ for every $i = 1, ..., t$ and $y_1, ..., y_t$ are all the intersection points that induce the Spin^c structure $\mathfrak{t}$.

Since each y_i is homogeneous and the monomial U drops the Alexander grading by one, we have that $\lambda_i(U) = 0$ for every $i = 1, ..., t$ and then $x = 0$.

2. Again the if implication is trivial. Now we have that

$$[x] = \sum_{i=1}^{r}[x_i] + [x]_T \,,$$

where $[x]_T$ is the projection of $[x]$ on the torsion submodule T. Since $F_*[x] = [0]$, it is

$$[x] = (1+U)[z] = (1+U)\lambda'_1(U)[z_1] + ... + (1+U)\lambda'_r(U)[z_r] + [x]_T \,;$$

where $[x_i] = (1+U)\lambda'_i(U)[z_i]$ for every $i = 1, ..., r$ and the $[z_i]'s$ are a homogeneous basis of the torsion-free quotient.

The same argument of 1 implies that the polynomials $\lambda'_i(U)$ are all zero and then $[x] = [x]_T$.

□

Now we use Lemma 5.4.2 to show that there is a correspondence between the torsion-free quotient of the link Floer homology group and the multi-pointed hat Heegaard Floer homology.

Theorem 5.4.3 *If L is an n-component link in a rational homology 3-sphere M then there exists an isomorphism of $\mathbb{F}[U]$-modules*

$$\frac{cHFL^-(M, L, \mathfrak{t} + \mathrm{PD}[L])}{T} \longrightarrow \left(\widehat{HF}(M, \mathfrak{t}) \otimes_{\mathbb{F}} \mathbb{F}[U]\right) \otimes_{\mathbb{F}[U]} \left(\mathbb{F}[U]_{(-1)} \oplus \mathbb{F}[U]_{(0)}\right)^{\otimes(n-1)} \;;$$

which sends a homology class of bigrading (d, s) into one of Maslov grading $d - 2s$.

Proof. We just have to show that F_* sends $\{[z_1], ..., [z_r]\}$, a homogeneous $\mathbb{F}[U]$-basis of the torsion-free quotient of $cHFL^-(M, L, \mathfrak{t} + \mathrm{PD}[L])$, into an $\mathbb{F}$-basis of $\mathcal{X} = \widehat{HF}(M, \mathfrak{t}) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)}$. Statement 1 in Lemma 5.4.2 tells us that F_* is surjective. In fact, if $[y] \in \mathcal{X}$ then it is $0 = \widehat{\partial}_z y = F(\partial^- x)$; where $F(x) = y$. We apply Lemma 5.4.2 to $\partial^-(x)$, since we can suppose that both x and y are homogeneous, and we obtain that $\partial^-(x) = 0$; then $[x]$ is a homology class. At this point, it is easy to see that the set $\{F_*[z_1], ..., F_*[z_r]\}$ is a system of generators of $\mathcal{X}$.

In order to prove that $F_*[z_1], ..., F_*[z_r]$ are also linearly independent in $\mathcal{X}$ we suppose that there is a subset $\left\{F_*[z_{i_1}], ..., F_*[z_{i_k}]\right\}$ such that

$$F_*[z_{i_1}] + ... + F_*[z_{i_k}] = F_*[z_{i_1} + ... + z_{i_k}] = [0] \, .$$

Then we apply Statement 2 of Lemma 5.4.2 to $[z_{i_1} + ... + z_{i_k}]$ and we obtain that it is a torsion class. This is a contradiction, because $[z_{i_1}], ..., [z_{i_k}]$ are part of an $\mathbb{F}[U]$-basis of a torsion-free $\mathbb{F}[U]$-module. $\square$

Chapter 6

Legendrian and contact invariants from open book decompositions

6.1 Legendrian Heegaard diagrams

6.1.1 Heegaard diagrams through abstract open books

In this chapter we introduce a specific type of Heegaard diagrams, obtained from adapted open book decompositions, that we call Legendrian Heegaard diagrams. Given an adapted open book decomposition (B, π, A), compatible with the triple (L, M, ξ), see Subsection 4.2.1, a *Legendrian Heegaard diagram* is a multi-pointed Heegaard diagram $D = (\Sigma, \alpha, \beta, w, z)$ which is obtained from the abstract open book associated to (B, π, A), defined in Subsection 4.2.2.

Denote such abstract open book with $(S_1, \Phi, \mathcal{A}, z, w)$. We have that $S_{\pm 1} = \overline{\pi^{-1}(\pm 1)}$. The sets of arcs in S_1 are $\mathbf{A} = \{a_1, ..., a_{2g+l+n-2}\}$ and $\mathbf{B} = \{b_1, ..., b_{2g+l+n-2}\}$, as in Figure

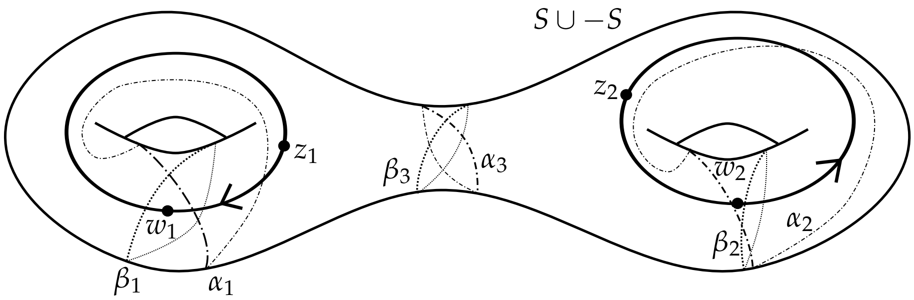


Figure 6.1: A diagram for the standard Legendrian 2-unlink in (S^3, ξ_{st}), obtained from an open book decomposition with page as in Figure 4.5.

4.8. We recall that g is the genus of S_1, while l is the number of its boundary components and n is the number of components of L. We define $\overline{a}_i$ and $\overline{b}_i$ for $i = 1, ..., 2g + l + n - 2$ as in Subsection 4.2.2.

Then D is gotten as follows:

- the surface Σ is $S_1 \cup -S_{-1}$; since π is a locally trivial fibration the pages S_1 and S_{-1} are diffeomorphic, but we reverse the orientation of the second one when we glue them together;

- the curves are given by $\alpha_i = b_i \cup \overline{b_i}$ and $\beta_i = a_i \cup \overline{a_i}$ for every $i = 1, ..., 2g + l + n - 2$;
- the z's and the w's are the set of baspoints that we introduced in Section 4.2.

In the settings of Chapter 4 the Legendrian Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ is a diagram for the (smooth) link L in the 3-manifold $-M$, see [72] and Subsection 4.2.2 before, that is M considered with the opposite orientation. We remark that here α and β are swapped with respect to the convention of Ozsváth and Szabó, which is the one we used in Chapter 4. See Figure 6.1 for an example.

We observe that, given (B, π, A), the only freedom in the choice of the Legendrian Heegaard diagram is in the arcs $\overline{b}_1, ..., \overline{b}_{2g+l+n-2}$ inside S_{-1}, which can be taken in the isotopy class of $\Phi(b_1), ...\Phi(b_{2g+l+n-2})$.

Proposition 6.1.1 *Suppose $D = (\Sigma, \alpha, \beta, w, z)$ is a Legendrian Heegaard diagram given by an adapted open book decomposition compatible with (L, M, ξ), where M is a rational homology 3-sphere. Then D is always admissible up to isotopy of the arcs in $\overline{\mathbf{B}}$.*

Proof. It follows from Lemma 5.1.5 and [77]. □

Our strategy is to study how Legendrian isotopic triples are related to one another, which means to define a finite set of moves between two adapted open book decompositions. Then find chain maps $\Psi_{(D_1,D_2)}$ for each move, that induce isomorphisms in homology, which will be particular cases of the maps in Section 5.3.

6.1.2 Global isotopies

We want to show that, given two Legendrian isotopic links $L_1, L_2 \hookrightarrow (M, \xi)$, two open book decompositions (B_i, π_i, A_i), compatible with the triples (L_i, M, ξ), are related by a finite sequence of moves. The first lemma follows easily.

Lemma 6.1.2 *Let us consider an adapted open book decomposition (B_1, π_1, A_1), compatible with the triple (L_1, M, ξ), and suppose that there is a contact isotopy of (M, ξ), sending L_1 into L_2.*

Then the time-1 map of the isotopy is a diffeomorphism $F : M \to M$ such that $\left(F(B_1), \pi_1 \circ F^{-1}, F(A_1)\right)$ is an adapted open book decomposition, compatible with (L_2, M, ξ).

Proof. The proof of the statement can be checked without effort. □

This lemma says that, up to global contact isotopies, we can consider (B_i, π_i, A_i) to be both compatible with a triple (L, M, ξ), where the link L is Legendrian isotopic to L_i for $i = 1, 2$. In other words, we can just study the relation between two open book decompositions compatible with a single triple (L, M, ξ).

6.1.3 Positive stabilizations

Let us start with a pair (S, Φ). A positive stabilization of (S, Φ) is the pair $\left(\widetilde{S}, \widetilde{\Phi}\right)$ obtained in the following way:

- the surface $\widetilde{S}$ is given by adding a 1-handle H to S;
- the monodromy $\widetilde{\Phi}$ is isotopic to $\Phi' \circ D_\gamma$. The map Φ' concides with Φ on S and it is the Identity on H. While D_γ is the right-handed Dehn twist along a curve γ; which intersects $S \cap H$ transversely precisely in the attaching sphere of H. See Figure 6.2.

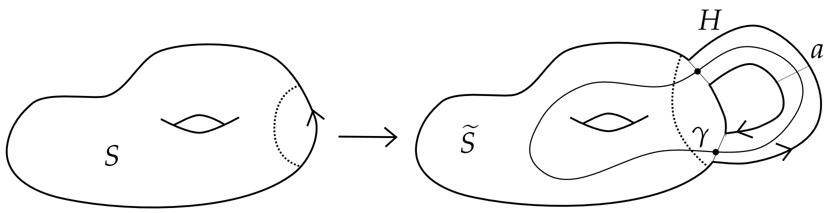


Figure 6.2: The arc $\gamma \cap \mathring{S}$ is a generic arc in the interior of S.

We say that (B', π', A') is a *positive stabilization* of (B, π, A) if

- The pair (S', Φ'), obtained from (B', π'), is a positive stabilization of (S, Φ), the one coming from (B, π).
- The system of generators A' is isotopic to $A \cup \{a\}$, where a is the cocore of H as in Figure 6.2.

We call a positive stabilization *L-elementary* if the curve γ as above, see Figure 6.2, intersects the link L, that sits in S and then also in $\widetilde{S}$, in at most one point transversely. Then we prove the following theorem, a generalization of a result of Giroux [39]. More details can be found in [27].

Theorem 6.1.3 (Giroux) *If (B_i, π_i) are two open book decompositions, compatible with the triple (L, M, ξ), then they admit isotopic L-elementary stabilizations. Namely, there is another compatible open book (B, π) which is isotopic to both (B_1, π_1) and (B_2, π_2), after an appropriate sequence of L-elementary stabilizations.*

The proof of Theorem 6.1.3 requires many steps. We recall that the definitions of contact cell decomposition and ribbon of a Legendrian graph can be found in Subsection 4.2.1.

Lemma 6.1.4 *Every open book compatible with (L, M, ξ), after possibly positively stabilizing, comes from a contact cell decomposition.*

Proof. Let S be the page of the open book decomposition and G be the core of S. That is, G is a graph embedded in S onto which S retracts. From an observation in [27] we can Legendrian realize G. Therefore, we note that S is the ribbon of G. Let $\nu(S)$ be a neighborhood of S such that $\partial S \subset \partial \nu(S)$.

We now follow the proof in [27]. Let a_i be a collection of properly embedded arcs on S that cut S into a disk. Let A_i be $a_i \times [0,1]$ in $M \setminus S = S \times [0,1]$ and $A_i' = A_i \cap \left(\overline{M \setminus \nu(S)}\right)$.

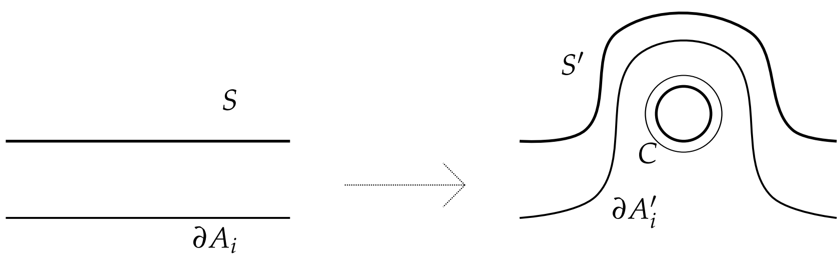


Figure 6.3: Abstract picture of the stabilization on S.

We observe that each A_i intersects ∂S on $\partial \nu(S)$ exactly twice. Thus, if we extend the A_i's into $\nu(S)$ so that their boundaries lie on G then the twisting of S, and hence ξ, along ∂A_i with respect to A_i is -1,0 or 1. If the value of all the twisting is -1 then we have a contact cell decomposition, because the contact structure restricted to the complement of S is tight. Then we just need to see how to reduce the twisting of ξ along ∂A_i.

Suppose ∂A_i has twisting 0. Then positively stabilize S as shown in Figure 6.3. Note that the curve C can be assumed to be Legendrian and bounds a disk E in M. Now isotope G across E to get a new Legendrian graph with all the A_j's unchanged except that A_i is replace with the disk A_i' obtained from A_i by isotoping across E. We also add C to G and add E to the 2-skeleton.

Clearly the twisting of ξ along E is -1. Thus we can reduce the twisting of ξ along ∂A_i as needed and after sufficiently many positive stabilizations we have an open book that comes from a contact cell decomposition. □

We observe that, in the proof of Lemma 6.1.4, we only use L-elementary stabilizations. We apply Lemma 6.1.4 to both (B_i, π_i). Obviously, we can also suppose that the 1-skeleton of (M, ξ) that we have in the beginning is a subgraph of the one after the stabilizations. At this point, we need to know how to relate two different contact cell decompositions for the same triple (L, M, ξ). To do this we use the following proposition.

Proposition 6.1.5 *Given two contact cell decompositions, for the same contact manifold, we have that they are related by a finite sequence of the following moves:*

1. *a subdivision of a 2-cell by a Legendrian arc, intersecting the dividing set in one point;*

2. *add a 1-cell c' and a 2-cell E so that $\partial E = c' \cup c$; where c is part of the original 1-skeleton and* $\mathrm{tb}(\partial E) = -1$*;*

3. *add a 2-cell E whose boundary is already in the 1-skeleton and* $\mathrm{tb}(\partial E) = -1$.

Proof. The proof follows from standard results in contact topology. □

The definition of dividing set of a convex surface is in Subsection 3.1.1. Clearly, we

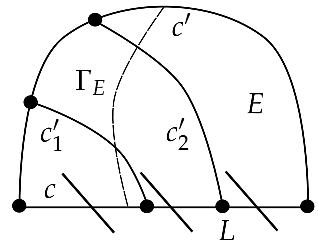


Figure 6.4: The arc Γ_E is the dividing set of the 2-cell E.

have that Move 1 corresponds to an L-elementary stabilization of (B_1, π_1), while Move 3 does not change the open book at all. For Move 2 on the other hand we have to reason more. In fact it can happen that the arc c in the original 1-skeleton intersects the link L transversely in more than one point. We solve this problem by applying other appropriate Moves 1 as in Figure 6.4. We observe that we obtain the same contact cell decomposition if we apply a sequence of Moves 2, using the arcs c_i', which now will give L-elementary stabilizations. To complete the proof we only need the following lemma.

Lemma 6.1.6 *Suppose (B_i, π_i) are two open books, compatible with (L, M, ξ), which come from the same contact cell decomposition. Then we have that (B_1, π_1) is isotopic to (B_2, π_2).*

Proof. First we have that the pages $\overline{\pi_1^{-1}(1)}$ and $\overline{\pi_2^{-1}(1)}$ are isotopic, because it is easy to see that the ribbon of the 1-skeleton is unique up to isotopy. Then we can suppose that the bindings coincide and $S = \overline{\pi_i^{-1}(1)}$ for $i = 1, 2$. Now, we have that $M \setminus S$ fibers along the interval $(0, 1)$ in two different ways. At this point, since $(0, 1)$ is contractible the two fibrations result to be trivial and then isotopic. It remains to see what happens when we glue S back.

The two open books induce monodromies Φ_1 and Φ_2 on S and they are conjugate because otherwise, from Corollary 4.2.3, the resulting open books would be compatible with non contactomorphic contact manifolds; but this is impossible. Then the fact that the monodromies are in the same conjugacy class tells us precisely that we can extend the isotopy on S and this completes the proof. $\square$

In other words, we may need to stabilize both open book decompositions many times and eventually we obtain other two open book decompositions $\left(B, \pi, \widetilde{A}_i\right)$, both compatible with $L \hookrightarrow (M, \xi)$; which is contact isotopic to (L_i, M, ξ_i) for $i = 1, 2$.

6.1.4 Admissible arc slides

In this subsection we use the terminology introduced in Subsection 4.2.1. Take an adapted system of generators A for an n-component link L, lying inside a surface S. We define admissible arc slide, a move that change $A = \{a_1, ..., a_i, ..., a_j, a_{2g+l+k-2}\}$ into

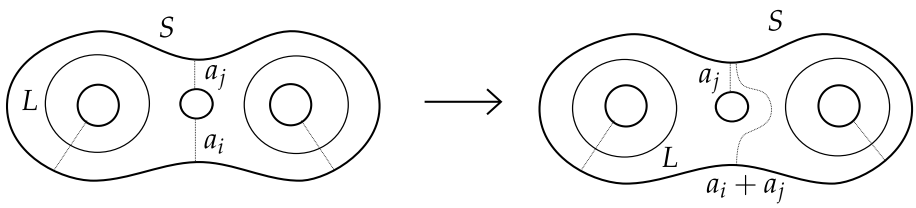


Figure 6.5: The arc $a_i + a_j$ is obtained by sliding a_j over a_i.

$A' = \{..., a_i + a_j, ..., a_j, ...\}$; where a_j is not a distinguished arc and one of the endpoints of a_i and a_j are adjacent, like in Figure 6.5. We can prove the following proposition.

Proposition 6.1.7 *Let us consider two open book decompositions (B, π, A_i), compatible with the Legendrian link $L \hookrightarrow (M, \xi)$. Then, after a finite number of admissible arc slides and isotopies on A_i, the open books coincide.*

We need two preliminary lemmas.

Lemma 6.1.8 *An admissible arc slide, from A to A', can be inverted. In the sense that we can perform another admissible arc slide, now from A' to A'', such that A'' is isotopic to A.*

Proof. If the arc slide changes the arc a_i into $a_i + a_j$ then it is easy to see that we can just slide $a_i + a_j$ over an arc a_j', isotopic to a_j; in a way that $a_i + a_j + a_j'$ is isotopic to a_i. $\square$

Lemma 6.1.9 *Suppose that we perform an admissible arc slide that changes a_i into $a_i + a_j$. Then we have the following facts:*

a) The set $\{a_1, ..., a_{2g+l-1}\}$ is a basis of $H_1(S, \partial S; \mathbb{F})$ and $i \leqslant 2g + l - 1$ if and only if $\{a_1, ..., a_i + a_j, ..., a_{2g+l-1}\}$ is a basis of $H_1(S, \partial S; \mathbb{F})$,

b) The arc a_i is distinguished if and only if $a_i + a_j$ is a distinguished arc,

c) The arc a_i is separating if and only if $a_i + a_j$ is a separating arc,

d) The arc a_i is dead if and only if $a_i + a_j$ is a dead arc.

Proof. a) Obviously, the arc $a_i + a_j$ represents the sum of the relative homology classes of a_i and a_j. At this point we just need basic linear algebra.

b) This claim is trivial.

c) We take $\{a_1, ..., a_{2g+l-1}\}$ as a basis of the first relative homology group of S. We can suppose that $i \geqslant 2g + l$. This is because if a_i is the only separating arc for a component of $S \setminus \{a_1, ..., a_{2g+l-1}\}$ then a_i is homologically trivial; otherwise, we can switch a_i with another separating arc for the same component.

   Now, if $a_i + a_j$ is not a separating arc then $S \setminus \{a_1, ..., a_{2g+l-1}, a_i + a_j\}$ should remain connected, but this is impossible because $\{a_1, ..., a_{2g+l-1}\}$ is a basis.

   The other implication follows easily from Lemma 6.1.8.

d) It follows from $b)$ and $c)$.

□

The strategy of the proof of Proposition 6.1.7 is, say $S_1, ..., S_n$ and $S'_1, ..., S'_n$ are the components of S minus the separating arcs of A_1 and A_2 respectively, we modify all the separating arcs, in both A_1 and A_2, with admissible arc slides; in order for S_i to concide with S'_i for every $i = 1, ..., n$. Moreover, we also want that each separating arc in A_1 becomes isotopic to a separating arc in A_2.

At the end, the components of S minus the separating arcs will be n surfaces, each one containing a component of L, a distinguished arc and the same number of dead arcs. We

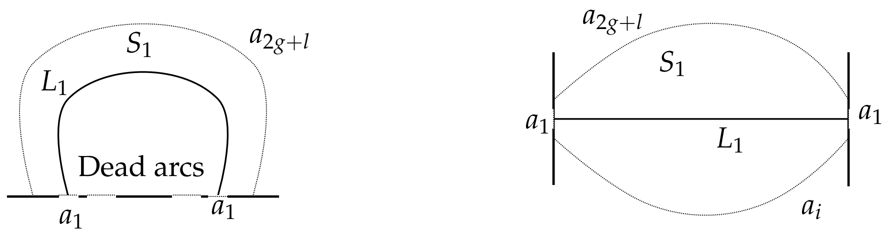


Figure 6.6: Case 1 is on the left. The figure shows a portion of $S \setminus A_1$.

conclude applying n times a result in [57], which proves the claim in the case of Legendrian knots.

Proof of Proposition 6.1.7. We want to prove that there is an adapted system of generators A for L in $S_1 = \overline{\pi^{-1}(1)}$ such that A is obtained, from A_1 and A_2, by a sequence of admissible arc slides (and isotopies).

We start from the component S_1. We can suppose that ∂S_1 contains the separating arc $a_{2g+l} \subset A_1^3$, the set of separating arcs in $\subset A_1$, and the distinguished arc $a_1 \subset A_1^1$, the set of distinguished arcs in A_1, with almost adjacent endpoints. We can have two cases: in the first one there are no other separating arcs in the portion of ∂S_1 on the opposite side of a_{2g+l}. In the second case they appear; possibly more than one, but we can suppose that there is exactly one of them. See Figure 6.6.

Case 1. When we consider S_1', which contains the same component of L that is in S_1, after some arc slides we have that it appears as in Figure 6.6 (left). This is because in the

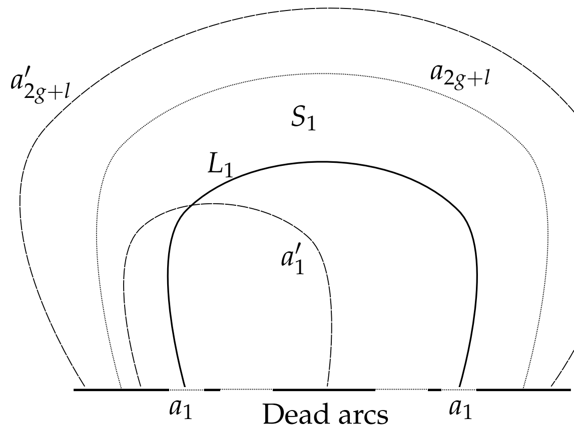


Figure 6.7: There are no other separating arcs, except for a_{2g+l} in S_1.

same figure we see that S_1 is split in two pieces by L_1 and the innermost one is not connected in any way to other components of S; in fact there are no separating arcs on that side. This means that the same holds for S_1' too. At this point it is easy to see that S_1 can be modified to be like in Figure 6.7; more explicitly, the separating arcs are parallel and the distinguished arcs lie in the region where L_1 is.

Case 2. As before we have that also S_1' appears like in the right part of Figure 6.6 (always after some arc slides). The reason is the same of previous case. Hence, now we can modify S_1 to be like in Figure 6.8; just in the same way as we did in Case 1.

We have obtained that the separating arcs are fixed on S_1 and S_1' and then the surfaces now have isotopic boundaries. Hence, we can move to another component S_2, whose

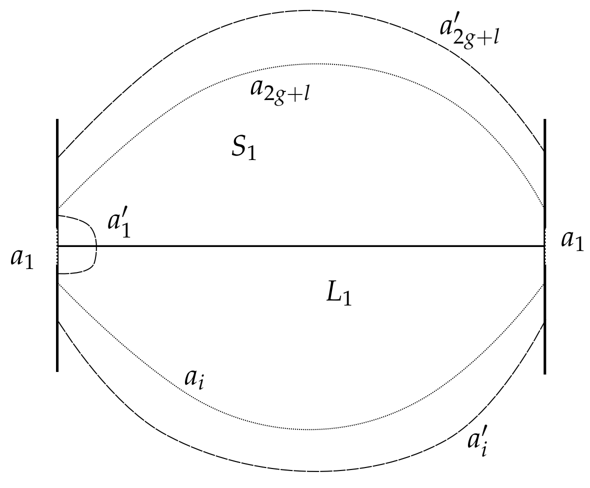


Figure 6.8: There are exactly two separating arcs, namely a_{2g+l} and a_i in S_1.

boundary contains a separating arc that has not yet been fixed, and we repeat the same process described before. We may need to slide some separating arcs over the ones that we already fixed in the previous step, but this is not a problem. We just iterate this procedure until all the S_i coincides with S_i' and this completes the proof. □

The results in this section prove the following theorem.

Theorem 6.1.10 *If $L_1, L_2 \hookrightarrow (M, \xi)$ are Legendrian isotopic links then the open book decompositions (B_i, π_i, A_i), which are compatible with the corresponding triples, are related by a finite sequence of global contact isotopies, positive stabilizations and admissible arc slides.*

Proof. It follows immediately from Propositions 6.1.1 and 6.1.7, Lemma 6.1.2 and Theorem 6.1.3. □

Though they are easy to deal with, we must not forget that, when we define the corresponding Legendrian Heegaard diagrams, we need to consider the choices of the monodromy and the families of arcs and basepoints inside their isotopy classes.

6.2 The Legendrian and transverse invariants

In this subsection we identify a cycle in the link Floer chain complexes that we previously introduced. The corresponding homology class will be our Legendrian invariant. Let us consider the only intersection point of $D = (\Sigma, \alpha, \beta, w, z)$ which lies in the page S_1. We recall that, in general, the intersection points live in the space $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, but they can be represented inside Σ. We denote this element with $\mathfrak{L}(D)$.

Proposition 6.2.1 *The intersection point $\mathfrak{L}(D)$ is such that $\partial^- \mathfrak{L}(D) = 0$ and then $\mathfrak{L}(D)$ is a cycle in $cCFL^-\left(D, \mathfrak{t}_{\mathfrak{L}(D)}\right)$; where $\mathfrak{t}_{\mathfrak{L}(D)}$ is the Spin^c structure that it induces on $-M$.*

Proof. Every $\phi \in \pi_2(\mathfrak{L}(D), y)$ such that $\mu(\phi) = 1$, where $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, has the property that $n_z(\phi) > 0$. The claim follows easily from the definition of the differential. □

At the end of the chapter we show that more can be said on the Spin^c structure $\mathfrak{t}_{\mathfrak{L}(D)}$. Now we want to show that the isomorphism class of the element $[\mathfrak{L}(D)]$ inside the homology group $cHFL^-\left(-M, L, \mathfrak{t}_{\mathfrak{L}(D)}\right)$ is a Legendrian invariant of the triple (L, M, ξ).

Let us be more specific: we consider a Legendrian link $L \hookrightarrow (M, \xi)$ in a rational homology contact 3-sphere; we associate two open book decompositions compatible with (L, M, ξ), say (B_1, π_1, A_1) and (B_2, π_2, A_2), and these determine (up to isotopy) two Legendrian Heegaard diagrams, that we call $D_1 = (\Sigma_1, \alpha_1, \beta_1, w_1, z_1)$ and $D_2 = (\Sigma_2, \alpha_2, \beta_2, w_2, z_2)$ respectively. Then we want to find a chain map

$$\Psi_{(D_1, D_2)} : cCFL^-(D_1, \mathfrak{t}) \longrightarrow cCFL^-(D_2, \mathfrak{t}), \tag{6.1}$$

that induces an isomorphism in homology, preserves the bigrading and it is such that

$$\Psi_{(D_1, D_2)}(\mathfrak{L}(D_1)) = \mathfrak{L}(D_2),$$

where $\mathfrak{t} = \mathfrak{t}_{\mathfrak{L}(D_1)} = \mathfrak{t}_{\mathfrak{L}(D_2)} \in \text{Spin}^c(-M)$. This is done in order to prove the following theorem.

Theorem 6.2.2 *Let us consider a Legendrian Heegaard diagram D, given by an open book compatible with a triple (L, M, ξ), where M is a rational homology 3-sphere and L is an n-component Legendrian link.*

Let us take the cycle $\mathfrak{L}(D) \in cCFL^-(D, \mathfrak{t}_{\mathfrak{L}(D)})$. Then, the isomorphism class of $[\mathfrak{L}(D)]$ in $cHFL^-(-M, L, \mathfrak{t}_\xi)$ is a Legendrian invariant of (L, M, ξ) and we denote it with $\mathfrak{L}(L, M, \xi)$. Furthermore, the Spin^c structure $\mathfrak{t}_{\mathfrak{L}(D)}$ coincides with $\mathfrak{t}_\xi + \mathrm{PD}[L]$.

Here we denote with $\mathfrak{t}_\xi$ the Spin^c structure induced on M by ξ.

Proposition 6.2.3 *Given a contact 3-manifold (Y, ζ) we have that ζ induces a Spin^c structure $\mathfrak{t}_\zeta$ on Y.*

Proof. The contact structure ζ gives a cooriented 2-plane field π on Y. Hence, if we take the isotopy class of the vector field orthogonal to π and oriented coherently with Y then, according to Turaev definition, we obtain a Spin^c structure on Y that we denote with $\mathfrak{t}_\zeta$. $\square$

In Section 5.1 we define another chain complex $\left(\widehat{CFL}(D, \mathfrak{t}), \widehat{\partial}\right)$ for every Spin^c structure $\mathfrak{t}$ on $-M$, whose homology is

$$\widehat{HFL}(-M, L, \mathfrak{t}) \, .$$

The intersection point $\mathfrak{L}(D)$ is a cycle also in $\widehat{CFL}(D, \mathfrak{t})$ and it also determines the Spin^c structure $\mathfrak{t}_{\mathfrak{L}(D)}$. Then the same procedure will show that the homology class $[\mathfrak{L}(D)]$ in $\widehat{HFL}(-M, L, \mathfrak{t})$ is also a Legendrian invariant of L.

Theorem 6.2.4 *The isomorphism class of $[\mathfrak{L}(D)]$ in $\widehat{HFL}\left(-M, L, \mathfrak{t}_\xi + \mathrm{PD}[L]\right)$ is a Legendrian invariant of (L, M, ξ) and we denote it with $\widehat{\mathfrak{L}}(L, M, \xi)$. Moreover, if $\widehat{\mathfrak{L}}(L, M, \xi)$ is non-zero then $\mathfrak{L}(L, M, \xi)$ is also non-zero.*

We recall that in Section 3.3 we give the definition of the transverse push-off T_L of a Legendrian link L. In the same way, if T is a transverse link then we can define a Legendrian approximation L_T of T. The procedure is described in [26]. The Legendrian link L_T is not well-defined up to Legendrian isotopy, but only up to negative stabilizations, see Section 3.5. Then we can also define transverse invariants from these link Floer homology groups by taking

$$\mathfrak{T}(T, M, \xi) = \mathfrak{L}(L_T, M, \xi) \quad \text{and} \quad \widehat{\mathfrak{T}}(T, M, \xi) = \widehat{\mathfrak{L}}(L_T, M, \xi) \, ,$$

where L_T is a Legendrian approximation of T.

Theorem 6.2.5 *The isomorphism classes $\mathfrak{T}(T, M, \xi)$ in $cHFL^-\left(-M, T, \mathfrak{t}_\xi + \mathrm{PD}[T]\right)$, and $\widehat{\mathfrak{T}}(T, M, \xi)$ in $\widehat{HFL}\left(-M, T, \mathfrak{t}_\xi + \mathrm{PD}[T]\right)$, are transverse link invariants. Furthermore, we have that $\widehat{\mathfrak{T}}(T, M, \xi)$ has the same non-vanishing property of $\widehat{\mathfrak{L}}$.*

In the case of knots the invariants $\mathfrak{L}$ and $\mathfrak{T}$ have been introduced first in [57].

6.3 Proof of the invariance

6.3.1 Global isotopies and admissible arc slides

If two open book decompositions are related by a global isotopy then it easy to see that the induced abstract open book coincide, up to conjugation of the monodromy and isotopy of the curves and the basepoints in the diagrams.

Hence, let us consider an abstract open book $(S, \Phi, \mathcal{A}, z, w)$ and recall that in S we have two set of arcs A and B, as explained in Subsection 4.2.2. The first check is easy: in fact if we perturb the basepoints inside the corresponding components of $S \setminus (A \cup B)$ then even the complex $(cCFL^-(D, \mathfrak{t}), \partial^-)$ does not change; where D is a Legendrian Heegaard diagram obtained from $(S, \Phi, \mathcal{A}, z, w)$. The same is true for an isotopy of S.

Now for what it concerns S we do not have problems, but when we define the chain complex we also consider the closed surface Σ, obtained by gluing together S and $-S$. We still have some choices on $-S$, in fact by Proposition 6.1.1 we may need to modify the arcs $\overline{B}$ in their isotopy classes to achieve admissibility for $D = (\Sigma, \alpha, \beta, w, z)$. Then the proof rests on the following proposition.

Proposition 6.3.1 *Suppose that two curves in a Heegaard diagram are related by the move shown in Figure 5.3. Then we can find a map $\Psi_{(D_1,D_2)}$ as in Equation* (6.1).

Proof. The map $\Psi_{(D_1,D_2)}$ is constructed using a Hamiltonian diffeomorphism of the surface, as described in Subsection 5.3.2. Since the new disks appear in $-S$, we have that $\mathfrak{L}(D_1)$ is sent to $\mathfrak{L}(D_2)$ which both lie in S. $\square$

An arc slide $\{..., a_i, ..., a_j, ...\} \to \{..., a_i + a_j, ..., a_j, ...\}$ in S corresponds to a handleslide $\{..., \alpha_i, ..., \alpha_j, ...\} \to \{..., \alpha_i', ..., \alpha_j, ...\}$ in Σ, where $\alpha_i' = a_i + a_j \cup \overline{a_i + a_j} \subset \Sigma$. Thus, a chain map $\Psi_{(D_1,D_2)}$, which induces an isomorphism in homology, is obtained by counting holomorphic triangles, as explained in Subsection 5.3.3. The admissibility of the arc slide is required only to avoid crossing a basepoint in w.

Remember that for every arc slide we actually have two handleslides. In fact we need to slide both the α and the β curves.

The fact that $\Psi_{(D_1,D_2)}(\mathfrak{L}(D_1)) = \mathfrak{L}(D_2)$ follows from [47]; where the arc slides invariance is proved in the open books setting.

6.3.2 Invariance under positive stabilizations

At this point, in order to complete the proof of the invariance of $[\mathfrak{L}(D)]$, it would be enough to define $\Psi_{(D,D^+)}$, such that $\Psi_{(D,D^+)}(\mathfrak{L}(D)) = \mathfrak{L}(D^+)$, in the case where D and D^+ are obtained from an open book and one of its positive stabilizations.

Since we have already proved invariance under admissible arc slides, we can complete the open books (B, π) and (B^+, π^+) with every possible adapted system of generators and then eventually define the map $\Psi_{(D,D^+)}$.

Proposition 6.3.2 *Let us call $S = S_{g,l} = \overline{\pi^{-1}(1)} \supset L$ the page over 1 of (B, π). Then we can always find A, an adapted system of generators for L in S, with the property that A is disjoint from the arc $\gamma' = \gamma \cap S$; where γ is the curve that we used to perform the L-elementary stabilization.*

Proof. We have to study four cases:

a) The arc γ' intersects L (transversely in one point),

b) The intersection $\gamma' \cap L$ is empty and γ' does not disconnect S,

c) The intersection $\gamma' \cap L$ is empty, the arc γ' disconnects S and L is not contained in one of the two resulting connected components of S,

d) The intersection $\gamma' \cap L$ is empty, the arc γ' disconnects S and the link L lies in one of the resulting connected components of S.

Let us start with Case a). We observe that γ' does not disconnect S; in fact if this was the case then a component of L would be split inside the two resulting components of S, thus we would have at least another intersection point between L and γ'; which is forbidden. We define $A = \{a_1, ..., a_{2g+l+n-2}\}$ as follows. Take a_1 as a push-off of γ'; clearly a_1 is a distinguished arc, because it intersects L_1, a component of L, in one point. Now call a' the arc given by taking a push-off of L_1 and extend it through a_1, on one side of L_1; this is the same procedure described in the proof of Theorem 4.2.1. If a' disconnects S into S_1, S_2 and there are components of L that lie in both S_i then we take a' as a separating arc; thus we extend $\{a_1, a'\}$ to an adapted system of generators A by using Theorem 4.2.1. On the other hand if L is contained in S_2 and S_1 is empty then we consider $\{a_1, a''\}$; where a'' is another push-off of L_1, this time extended through a_1 on the other side of L_1. We still have problems when a' does not disconnect S. We can fix this by taking $\{a_1, a', a''\}$; which together disconnect S into two connected components, one of them containing only L_1 and the other one $L \setminus L_1$. Again we extend the set $\{a_1, a', a''\}$ to A applying Theorem 4.2.1.

The other three cases are easier. In Case b) we just denote with a_{n+1} the push-off of γ' and we complete it to A.

In Case c) we denote with a_{2g+l} our push-off of γ' and we take it as a separating arc. Then we can complete it to A.

Finally, Case d) is as follows. Since in this case the push-off is trivial in homology, and it bounds a surface disjoint from L, we can actually ignore it and easily find a set A which never intersects γ'. $\square$

Now we have the abstract open book $(S, \Phi, \mathcal{A}, z, w)$, obtained from (B, π, A). Denote with $(S^+, \Phi^+, \mathcal{A}^+, z, w)$ the one coming from (B^+, π^+, A^+), where $S^+ = \overline{(\pi^+)^{-1}(1)}$ is S with a 1-handle attached; $\Phi^+ = \Phi' \circ D_\gamma$ where Φ' coincides with Φ on S, extended with the Identity on the new 1-handle; and $A^+ = A \cup \{a\}$ with a being the cocore of the new 1-handle (see Figure 6.2). Then we call D and D^+ the corresponding Legendrian Heegaard diagrams.

We define $\Psi_{(D,D^+)}$ in the following way. For every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}_{\mathfrak{L}(D)}}$ it is

$$\Psi_{(D,D^+)}(x) = x' \, ;$$

where $x' = x \cup \{a \cap b\}$ with b being the arc in strip position with a, as in Figure 4.8.

It results that $\Psi_{(D,D^+)}$ is a chain map because the curve $\alpha = b \cup \overline{b}$ only intersects the curve $\beta = a \cup \overline{a}$, and moreover it is $\alpha \cap \beta = \{1 \text{ pt}\}$, since we choose A^+ in a way that every arc in it is disjoint from γ'. We have that $\Psi_{(D,D^+)}$ induces an isomorphism in homology, because it is an isomorphism also on the level of chain complexes, and sends $\mathfrak{L}(D)$ into $\mathfrak{L}(D^+)$.

6.3.3 Invariance theorems

We prove our main results in this chapter, which shows the invariance of $\mathfrak{L}$ and $\mathfrak{T}$. The claim about the Spin^c structure $\mathfrak{t}_{\mathfrak{L}(D)}$ is studied in Section 6.4.

Proof of Theorem 6.2.2. We proved that $\mathfrak{L}(D)$ is a cycle in Proposition 6.2.1. The invariance follows from the results obtained in this section: in fact, we proved that if D_1 and D_2 are Legendrian Heegaard diagrams, representing Legendrian isotopic links, then we have a chain map $\Psi_{(D_1,D_2)}$ that preserves the bigrading and the Spin^c structure and sends $\mathfrak{L}(D_1)$ to $\mathfrak{L}(D_2)$. $\square$

We note that the invariant can be a U-torsion class in the group $cHFL^{-}\left(-M, L, \mathfrak{t}_{\mathfrak{L}(D)}\right)$, which means that there is a $k \geqslant 0$ such that $U^k \cdot \mathfrak{L}(L, M, \xi) = [0]$. Moreover, the cycle $\mathfrak{L}(D)$ possesses a bigrading (d, s); such bigrading is induced on the invariant $\mathfrak{L}(L, M, \xi)$, because all the maps Ψ defined in this section preserves both the Maslov and the Alexander grading.

Proof of Theorem 6.2.4. The proof of this theorem is the same as the one of Theorems 6.2.2 and 6.2.5, except for the non-vanishing property, which follows from the fact that $\widehat{CFL}(D, \mathfrak{t}_{\mathfrak{L}(D)})$ is a quotient of $cCFL^{-}(D, \mathfrak{t}_{\mathfrak{L}(D)})$. □

6.4 A remark on the Ozsváth-Szabó contact invariant

Now we spend a few words about the Ozsváth-Szabó contact invariant $c(\xi)$, introduced in [70].

Given a Legendrian Heegaard diagram D. Let us call $c(D)$ the only intersection point which lies on the page S_1 as before, but now considered as an element in $\left(\widehat{CF}(D, \mathfrak{t}_{c(D)}), \widehat{\partial}_z\right)$. The proof of Proposition 6.2.1 says that $c(D)$ is also a cycle. Moreover, we have the following theorem.

Theorem 6.4.1 (Ozsváth and Szabó) *Let us consider a Legendrian Heegaard diagram D with a single basepoint z, given by an open book compatible with a pair (M, ξ), where M is a rational homology 3-sphere. Let us take the cycle $c(D) \in \widehat{CF}(D, \mathfrak{t}_{c(D)})$.*

Then the isomorphism class of $[c(D)] \in \widehat{HF}\left(-M, \mathfrak{t}_{c(D)}\right)$ is a contact invariant of (M, ξ) and we denote it with $\widehat{c}(M, \xi)$. Furthermore, we have the following properties:

- *the Spin^c structure $\mathfrak{t}_{c(D)}$ coincides with $\mathfrak{t}_\xi$;*
- *the Maslov grading of $c(D)$ is given by $M_z(c(D)) = -d_3(M, \xi)$.*

The proof of this theorem comes from [70], where Ozsváth and Szabó first define the invariant $\widehat{c}(M, \xi)$, and [47], where Honda, Kazez and Matić give the reformulation using open book decompositions that we use in this thesis.

Furthermore, now we can see which is the Spin^c structure induced by $\mathfrak{L}(D)$.

Corollary 6.4.2 *The intersection point $\mathfrak{L}(D)$ induces the Spin^c structure $\mathfrak{t}_\xi + \mathrm{PD}[L]$.*

This completes the proof of Theorem 6.2.2.

Proof of Corollary 6.4.2. We recall that

$$\mathfrak{s}_w(\mathfrak{L}(D)) - \mathfrak{s}_z(\mathfrak{L}(D)) = \mathrm{PD}[L]$$

from Proposition 4.1.3. Then we have that

$$\mathfrak{t}_{\mathfrak{L}(D)} = \mathfrak{s}_w(\mathfrak{L}(D)) = \mathfrak{s}_z(\mathfrak{L}(D)) + \mathrm{PD}[L] = \mathfrak{t}_{c(D)} + \mathrm{PD}[L] = \mathfrak{t}_\xi + \mathrm{PD}[L] \,,$$

from Theorem 6.4.1. □

We show that the argument used for the invariance of $\mathfrak{L}$ also proves a generalization of Theorem 6.4.1, which involves Legendrian Heegaard diagrams with more than one basepoint. This is done considering the fact that we can define the chain complex $\left(\widehat{CF}(D), \widehat{\partial}_z\right)$ starting from multi-pointed Heegaard diagrams; as we saw in Theorem 5.2.1.

Theorem 6.4.3 *Let us consider a Legendrian Heegaard diagram D, given by an open book compatible with a pair (M,ξ), where M is a rational homology 3-sphere. Let us take the cycle $c(D) \in \widehat{CF}(D, \mathfrak{t}_{c(D)})$.*

Then the isomorphism class of $[c(D)] \in \widehat{HF}\left(-M, \mathfrak{t}_{c(D)}\right) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)}$ is a contact invariant of (M,ξ) and we denote it with $\widehat{c}(M,\xi;n)$.

Proof. As in the proof of Theorem 6.1.10 we claim that two open book decompositions, compatible with contactomorphic manifolds (M_1,ξ_1) and (M_2,ξ_2), are related by a finite sequence of isotopies, arc slides and positive stabilizations.

First we observe that if (B_1,π_1,A_1) is the open book decomposition compatbile with (M_1,ξ_1) then $\big(F(M_1), F(\xi_2), F(A_1)\big)$ is compatible with (M_2,ξ_2), where F is the contactomorphism between the two manifolds.

At this point we suppose that (B_i,π_i) for $i = 1,2$ are compatible with a contact manifold (M,ξ). We argue that we can postively stabilize both many times until the two open book decompositions are isotopic to (B,π), which is also compatible with (M,ξ). This is done in the same way as in Subsection 6.1.3, using a weaker version of Giroux's Theorem 6.1.3.

We fix (B,π) compatible with (M,ξ). It only remains to show that, say A_1 and A_2 are two system of generators as in the definition of abstract open book in Subsection 4.2.2, there is a sequence of arc slides which changes A_1 into A_2. This follows by adapting the proof of Proposition 6.1.7.

At this point, we consider the Legendrian Heegaard diagrams D_1 and D_2, coming from open book decompositions compatible with contactomorphic manifolds. We define the chain maps $\Psi_{(D_1,D_2)}$ exactly as in the proof of Theorem 6.2.2; in fact, there are even less restrictions than before: the arc slides do not need to be admissible, because we do not consider the basepoints in w; moreover, the positive stabilizations do not need to be L-elementary, since there is no link on the page of (B,π). Then we conclude in the same way as in Theorem 6.2.2. $\square$

We observe that $\widehat{c}(M,\xi;n)$ is not a new invariant; in fact, we immediately have the following corollary.

Corollary 6.4.4 *Suppose that (M,ξ) is a rational homology contact 3-sphere. Then we have that*

$$\widehat{c}(M,\xi;n) = \widehat{c}(M,\xi) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)} .$$

Furthermore, if D is a Legendrian Heegaard diagram for (M,ξ) then we have the following properties:

- *the Spin^c structure induced by the cycle $c(D)$, representing $\widehat{c}(M,\xi;n)$, coincides with $\mathfrak{t}_\xi$;*
- *the Maslov grading of $c(D)$ is given by $M_z(c(D)) = -d_3(M,\xi) + 1 - n$.*

Proof. From Theorem 6.4.3 we find a chain map

$$\Psi_{(D,D_0)} : \left(\widehat{CF}(D), \mathfrak{t}\right) \longrightarrow \left(\widehat{CF}(D_0), \mathfrak{t}\right)$$

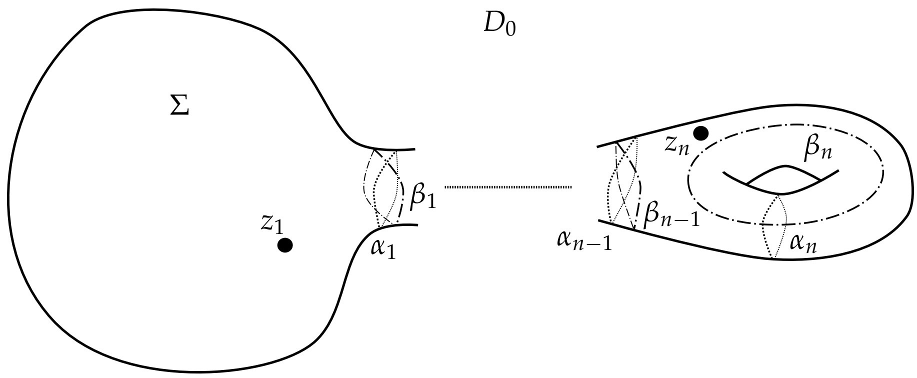


Figure 6.9: The Legendrian Heegaard diagram D_0 is obtained by stabilizing $n-1$ times D', a diagram compatible with (M, ξ).

which induces an isomorphism in homology, preserves the grading and sends $c(D)$ into $c(D_0)$, where D_0 is the Legendrian Heegaard diagram in Figure 6.9.

The diagram D_0 is also compatible with (M, ξ) because is gotten by stabilizing D', which is a single-pointed Legendrian Heegaard diagram taken from an open book decomposition compatible with (M, ξ).

Therefore, using Theorem 5.2.3, and the argument in its proof, we obtain that

$$c(D_0) = c(D') \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)} .$$

Then the first claim follows from Theorem 6.4.1.

The fact that $\mathfrak{t}$ coincides with $\mathfrak{t}_\xi$ is a direct consequence of Theorem 6.4.1, since stabilizing does not change the Spin^c structure. Finally, the cycle $c(D_0)$ has Maslov grading $c(D') + 1 - n$ because, at each stabilization, we add one coordinate in $c(D')$ which decrease its Maslov grading by one, as we saw in the proof of Lemma 5.2.2. This means that we can conclude again by using Theorem 6.4.1. □

6.5 Properties of $\mathfrak{L}$ and $\mathfrak{T}$

We recall that in Subsection 5.4 we introduce the surjective $\mathbb{F}$-linear map

$$F : cCFL^-(D, \mathfrak{t}_\xi + \mathrm{PD}[L]) \xrightarrow{U=1} \widehat{CF}(D^z, \mathfrak{t}_\xi)$$

which induces the map F_* in homology:

$$F_* : cHFL^-\left(-M, L, \mathfrak{t}_\xi + \mathrm{PD}[L]\right) \xrightarrow{U=1} \widehat{HF}\left(-M, \mathfrak{t}_\xi\right) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)} ,$$

where D is a Legendrian Heegaard diagram coming from an open book decomposition compatible with (M, ξ).

Then we can prove the following lemma.

Lemma 6.5.1 *The map F as above is such that $F_*(\mathfrak{L}(L, M, \xi)) = \widehat{c}(M, \xi) \otimes (\boldsymbol{e}_{-1})^{\otimes(n-1)}$, where $\boldsymbol{e}_{-1}$ is the generator of $\mathbb{F}_{(-1)}$.*

Proof. Since $\mathfrak{L}(D)$ and $c(D)$ both correspond to the unique intersection point on the page of the open book decomposition, we immediately see that $F(\mathfrak{L}(D)) = c(D)$. Then the claim follows from Theorems 6.2.2 and 6.4.3. $\square$

Lemma 6.5.1 implies the following important result.

Corollary 6.5.2 *The Legendrian invariant $\mathfrak{L}(L, M, \xi)$ is non-torsion if and only if the contact invariant $\widehat{c}(M, \xi)$ is non-zero.*

Furthermore, if D is a Legendrian Heegaard diagram compatible with (L, M, ξ) then the Maslov and Alexander grading of the element $\mathfrak{L}(D)$ are related by the following equality:

$$M(\mathfrak{L}(D)) = -d_3(M, \xi) + 2A(\mathfrak{L}(D)) + 1 - n\,;$$

where n is the number of component of L.

Proof. It follows from Lemma 5.4.1 and Theorem 6.4.3. $\square$

In particular, this corollary says that the Legendrian link invariant $\mathfrak{L}$ is always a torsion class if ξ is overtwisted and always non-torsion if (M, ξ) is symplectically fillable. In fact, from [70] we know that $\widehat{c}(M, \xi)$ is zero in the first case and non-zero in the second one.

We proved that the isomorphism class $\mathfrak{L}(L, M, \xi)$ is a Legendrian invariant. This is true also for the Alexander (and Maslov) grading of the element $\mathfrak{L}(D) \in cCFL^-(D, \mathfrak{t}_\xi)$, that we denote with $A(\mathfrak{L}(D))$.

From Corollary 6.5.2 we know that $\widehat{c}(M, \xi) \neq [0]$ implies that $\mathfrak{L}(L, M, \xi)$ is non-torsion, hence it determines $A(\mathfrak{L}(D))$. On the other hand, if $\widehat{c}(M, \xi)$ is zero then a priori the gradings of the element $\mathfrak{L}(D)$ could give more information.

Starting from these observations, we want to express the value of $A(\mathfrak{L}(D))$ in terms of the other known invariants of the Legendrian link L. Note that Corollary 6.5.2 also tells us that the Maslov grading of $\mathfrak{L}(D)$ is determined, once we know $A(\mathfrak{L}(D))$.

First we recall that, from the definitions of Thurston-Bennequin and rotation number in Chapter 3, we have

$$\mathrm{tb}(L) = \sum_{i=1}^{n} \mathrm{tb}_i(L)\,, \quad \text{where } \mathrm{tb}_i(L) = \mathrm{tb}(L_i) + \mathrm{lk}_{\mathbb{Q}}(L_i, L \setminus L_i)$$

and

$$\mathrm{rot}(L) = \sum_{i=1}^{n} \mathrm{rot}_i(L)\,, \quad \text{where } \mathrm{rot}_i(L) = \mathrm{rot}(L_i)\,.$$

Theorem 6.5.3 *Consider $L \hookrightarrow (M, \xi)$ an n-component Legendrian link in a rational homology contact 3-sphere and D a Legendrian Heegaard diagram, that comes from an open book decomposition compatible with (L, M, ξ). Then we have that*

$$A(\mathfrak{L}(D)) = \frac{\mathrm{tb}(L) - \mathrm{rot}(L) + n}{2} \quad \text{and} \quad M(\mathfrak{L}(D)) = -d_3(M, \xi) + \mathrm{tb}(L) - \mathrm{rot}(L) + 1\,.$$

Proof. If L is a null-homologous knot then the claim has been proved by Ozsváth and Stipsicz in [66]; moreover, the proof can be generalized to work with every knot in a rational homology 3-sphere.

At this point, in order to obtain the claim for links, we need to relate $A(\mathfrak{L}(D))$ with the Alexander grading of the Legendrian invariants of the components L_i of L.

A Legendrian Heegaard diagram D_i for the knot L_i is easily gotten from D by removing some curves and basepoints. We denote the intersection point representing the Legendrian invariant of L_i with $\mathfrak{L}(D_i)$. Then we have

$$\begin{aligned} A(\mathfrak{L}(D)) &= \sum_{i=1}^{n}\left(A\left(\mathfrak{L}(D_i)\right)+\frac{1}{2}\mathrm{lk}_{\mathbb{Q}}(L_i, L\setminus L_i)\right) = \\ &= \sum_{i=1}^{n}\left(\frac{\mathrm{tb}(L_i)-\mathrm{rot}(L_i)+1}{2}+\frac{\mathrm{lk}_{\mathbb{Q}}(L_i, L\setminus L_i)}{2}\right) = \sum_{i=1}^{n}\frac{\mathrm{tb}_i(L)-\mathrm{rot}_i(L)+1}{2} = \\ &= \frac{\mathrm{tb}(L)-\mathrm{rot}(L)+n}{2} \end{aligned}$$

and the claim follows. □

This property holds also for the invariant $\mathfrak{T}$ and we state it in the following corollary.

Corollary 6.5.4 *Suppose $T \hookrightarrow (M, \xi)$ is an n-component transverse link in a rational homology contact 3-sphere with $\widehat{c}(M, \xi) \neq [0]$. Then we have that the bigrading of $\mathfrak{T}$ is*

$$A(\mathfrak{T}(T, M, \xi)) = \frac{\mathrm{sl}(T)+n}{2} \quad \textit{and} \quad M(\mathfrak{T}(T, M, \xi)) = -d_3(M, \xi)+\mathrm{sl}(T)+1\,.$$

Proof. It follows from Theorems 6.2.5 and 6.5.3, Proposition 3.3.2 and Corollary 6.5.2. □

6.6 Connected sums and invariance of $\mathfrak{T}$

6.6.1 Behaviour of the invariants under connected sum

Let us consider two adapted open book decompositions (B_i, π_i, A_i), compatible with the triples (L_i, M_i, ξ_i). We can define a third open book (B, π, A), for the manifold $M_1 \# M_2$, with the property that $\pi^{-1}(1)$ is a Murasugi sum , see [27], of the pages $\pi_1^{-1}(1)$ and $\pi_2^{-1}(1)$, which is defined as follows.

Given two abstract open books $(S_i, \Phi_i, \mathcal{A}_i, z_i, w_i)$ for $i = 1, 2$, let c_i be an arc properly embedded in S_i and R_i a rectangular neighborhood of c_i such that $R_i = c_i \times [-1, 1]$. Then the *Murasugi sum* of (S_1, Φ_1) and (S_2, Φ_2) is the pair $(S_1, \Phi_1) * (S_2, \Phi_2)$ with page $S_1 * S_2 = S_1 \cup_{R_1=R_2} S_2$, where R_1 and R_2 are identified so that $c_i \times \{-1, 1\} = (\partial c_{i+1}) \times [-1, 1]$, and the monodromy is $\Phi_1 \circ \Phi_2$. Since we want to define an abstract open book, we also need to say what happens to the system of generators and the basepoints.

We may assume that suitable portions of the open books and the adapted system of generators appear as in the left in Figure 6.10. We take the Murasugi sum of the two open books in the domains containing the basepoints z_1 and w_1, which correspond to the components of L_1 and L_2 that we want to merge, and then we drop these two basepoints. We can make sure that the resulting oriented link $L_1 \# L_2$ is smoothly determined on a page of the resulting open book. The arcs can also be arranged to be the same as the ones illustrated in Figure 6.10. The resulting open book is compatible with the triple $(L_1 \# L_2, M_1 \# M_2, \xi_1 \# \xi_2)$.

Denote with D_1,D_2 and D the Legendrian Heegaard diagrams obtained from the open book decompositions that we introduced before. Then, we have the following theorem.

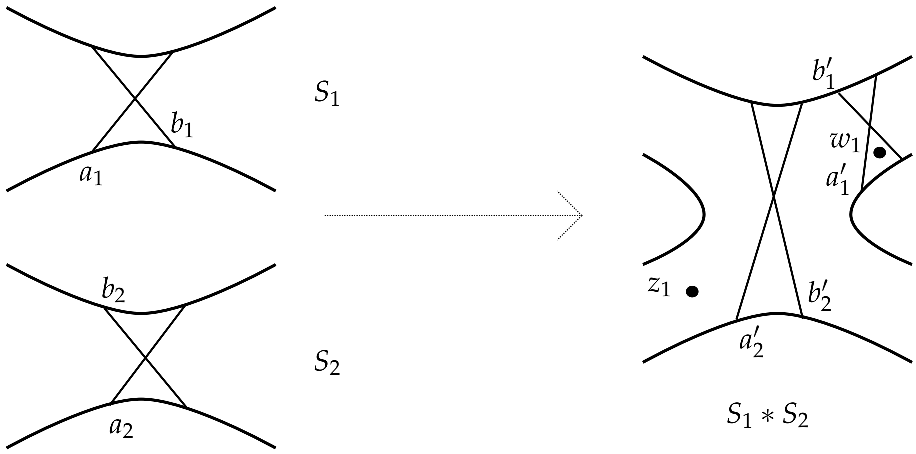


Figure 6.10: On the left we see the starting abstract open books, while on the right there is the Murasugi sum of the two.

Theorem 6.6.1 *For every Spin^c structure on M_1 and M_2 there is a chain map*

$$cCFL^-(D, \mathfrak{t}_1 \# \mathfrak{t}_2) \longrightarrow cCFL^-(D_1, \mathfrak{t}_1) \otimes_{\mathbb{F}[U]} cCFL^-(D_2, \mathfrak{t}_2)$$

that preserves the bigrading and the element $\mathfrak{L}(D)$ is sent into $\mathfrak{L}(D_1) \otimes \mathfrak{L}(D_2)$.

Furthermore, this map induces an isomorphism in homology, which means that

$$\begin{aligned} cHFL^- \left(-M_1 \# -M_2, L_1 \# L_2, \mathfrak{t}_{\xi_1 \# \xi_2} + \mathrm{PD}[L_1] + \mathrm{PD}[L_2]\right) \cong \\ cHFL^- \left(-M_1, L_1, \mathfrak{t}_{\xi_1} + \mathrm{PD}[L_1]\right) \otimes_{\mathbb{F}[U]} cHFL^- \left(-M_2, L_2, \mathfrak{t}_{\xi_2} + \mathrm{PD}[L_2]\right) \end{aligned}$$

as bigraded $\mathbb{F}[U]$-modules and

$$\mathfrak{L}(L_1 \# L_2, M_1 \# M_2, \xi_1 \# \xi_2) = \mathfrak{L}(L_1, M_1, \xi_1) \otimes \mathfrak{L}(L_2, M_2, \xi_2) .$$

Proof. We use the same strategy of the case of knots in [57]. There are three Heegaard

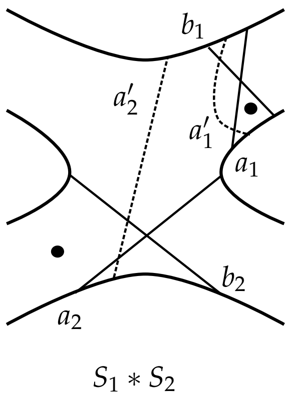


Figure 6.11: Intermediate situation between the slidings.

diagrams coming into play: the connected sum diagram, whose α and β-circles are gotten by doubling the initial bases, an intermediate diagram, shown in Figure 6.11, gotten by sliding the β's as dictated by the arcslides, whose attaching circles we denote β' and α, and the final one gotten by performing handleslides on the α-circles, as dictated by the arc slides of the b_i as in the right-most diagram in Figure 6.10, whose attaching cicles are β' and α'.

From [72] we know that there is a chain map

$$\Phi : cCFL^-(D_1) \otimes cCFL^-(D_2) \longrightarrow cCFL^-(D)$$

which induces an isomorphism in homology and preserves the bigrading and the Spin^c structure. In all three of these diagrams there is a unique intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ which is supported in $S_1 * S_2$: the first is $x = \Phi(\mathfrak{L}(D_1) \otimes \mathfrak{L}(D_2))$, the second one is denoted with x' and the third one, that we call x'', clearly determines the Legendrian invariant for $L_1 \# L_2$ and then coincides with $\mathfrak{L}(D)$.

We claim that the three generators are mapped to one another under the maps induced by handleslides. We argue first that x' is a cycle. We say that, for any $y \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\beta$, if $\phi \in \pi_2(x', y)$ has all non-negative local multiplicities, then $x' = y$ and and ϕ is the constant disk. We need to be careful to the fact that the arcs disconnect $S_1 * S_2$ into two regions, only one of which contains z_1. However, it is still easy to see from Figure 6.11 that any positive domain flowing out of x', which has positive multiplicity on the other region not containing z_1, has also to have positive multiplicity at z_1.

We now want to count holomorphic triangles. Consider the Heegaard triple $(\Sigma, \alpha, \beta, \beta', w, z)$, and the diagram $(\Sigma, \beta, \beta', z)$ which represents an unknot in the g-fold connected sum of $S^2 \times S^1$. Moreover, in that diagram there is a unique intersection point $\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_{\beta'}$, representing the top-dimensional Floer homology class, see also Subsection 5.3.3.

The handleslide map

$$\Psi_1 : cCFL^-(\Sigma, \alpha, \beta, w, z) \longrightarrow cCFL^-(\Sigma, \alpha, \beta', w, z)$$

is defined by

$$\Psi_1(x) = \sum_y \sum_{\substack{\psi \in \pi_2(x, \Theta, y) \\ \mu(\psi)=1, n_z(\psi)=0}} \left| \widehat{\mathcal{M}}(\psi) \right| \cdot U^{n_w(\psi)} y \, .$$

The claim that $\Psi_1(x) = x'$ follows from the following two facts:

- there is a triangle $\psi_0 \in \pi_2(x, \Theta, x')$, gotten as a disjoint union of the obvious small triangles in $S_1 * S_2$;
- every other triangle $\psi \in \pi_2(x, \Theta, y)$ with $n_z(\psi) = 0$ has a negative local multiplicity somewhere.

The first is obtained easily from Figure 6.11. To see the second, we notice that any homology class $\psi \in \pi_2(x', \Theta, y)$ can be decomposed as $\psi_0 * \phi$, where $\phi \in \pi_2(x', y)$. Moreover, if ψ has only positive local multiplicities, then the same follows for ϕ; also, since $n_z(\psi) = 0 = n_z(\psi_0)$, it follows that $n_z(\phi) = 0$ as well. But by our above argument that x' is a cycle, it now follows that ϕ is constant.

Consider next the handleslide map

$$\Psi_2 : (\Sigma, \alpha, \beta', w, z) \longrightarrow (\Sigma, \alpha', \beta', w, z) \, .$$

This is defined by pairing with an intersection point $\Theta' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\alpha$, representing the top-dimensional non-trivial homology. The argument that $\Psi_2(x') = x''$ follows through a similar argument.

Finally, we have that the composition $\overline{\Phi} = \Psi_2 \circ \Psi_1 \circ \Phi$ now induces an isomorphism on homology, carrying $\mathfrak{L}(D_1) \otimes \mathfrak{L}(D_2)$ into $x'' = \mathfrak{L}(D)$, which completes the proof of the theorem. $\square$

We note that the link Floer homology group of the connected sum does not depend on the choice of the components. In particular, this means that we can compute the link Floer homology group and the Legendrian invariant of a disjoint union $L_1 \sqcup L_2$, see Section 3.5.

Proposition 6.6.2 *If we denote with $\bigcirc_2$ a smooth 2-component unlink and with $\mathcal{O}_2$ the Legendrian 2-component unlink in (S^3, ξ_{st}) such that $\mathrm{tb}(\mathcal{O}_2) = -2$ then we have that*

$$cHFL^-\left(-M_1\# -M_2, L_1 \sqcup L_2, \mathfrak{t}_{\xi_1 \# \xi_2} + \mathrm{PD}[L_1] + \mathrm{PD}[L_2]\right) \cong$$
$$cHFL^-\left(-M_1, L_1, \mathfrak{t}_{\xi_1} + \mathrm{PD}[L_1]\right) \otimes_{\mathbb{F}[U]} cHFL^-\left(-M_2, L_2, \mathfrak{t}_{\xi_2} + \mathrm{PD}[L_2]\right) \otimes_{\mathbb{F}[U]} cHFL^-(\bigcirc_2)$$

and

$$\mathfrak{L}(L_1 \sqcup L_2, M, \xi) = \mathfrak{L}(L_1, M_1, \xi_1) \otimes \mathfrak{L}(L_2, M_2, \xi_2) \otimes \mathfrak{L}(\mathcal{O}_2) .$$

Proof. We just apply Theorem 6.6.1 twice, each time on one of the two components of $\mathcal{O}_2$. $\square$

The homology group $cHFL^-(\bigcirc_2)$ is isomorphic, as bigraded $\mathbb{F}[U]$-module, to $\mathbb{F}[U]_{(-1,0)} \oplus \mathbb{F}[U]_{(0,0)}$ and this can be proved immediately from the diagram in Figure 6.1. Moreover, Theorem 6.5.3 tells us that

$$\mathfrak{L}(\mathcal{O}_2) = \mathbf{e}_{-1,0} ,$$

that is the generator of $\mathbb{F}[U]$ with bigrading $(-1, 0)$. This means that, if $\widehat{c}(M, \xi)$ is non-zero, we have

$$M(\mathfrak{L}(L_1 \sqcup L_2, M, \xi)) = M(\mathfrak{L}(L_1, M_1, \xi_1)) + M(\mathfrak{L}(L_2, M_2, \xi_2)) - 1$$

and

$$A(\mathfrak{L}(L_1 \sqcup L_2, M, \xi)) = A(\mathfrak{L}(L_1, M_1, \xi_1)) + A(\mathfrak{L}(L_2, M_2, \xi_2)) .$$

It is easy to check that similar results hold also for the Legendrian invariant $\widehat{\mathfrak{L}}$.

Corollary 6.6.3 *We have that*

$$\widehat{\mathfrak{L}}(L_1\# L_2, M_1\# M_2, \xi_1\# \xi_2) = \widehat{\mathfrak{L}}(L_1, M_1, \xi_1) \otimes \widehat{\mathfrak{L}}(L_2, M_2, \xi_2)$$

and

$$\widehat{\mathfrak{L}}(L_1 \sqcup L_2, M, \xi) = \widehat{\mathfrak{L}}(L_1, M_1, \xi_1) \otimes \widehat{\mathfrak{L}}(L_2, M_2, \xi_2) \otimes \widehat{\mathfrak{L}}(\mathcal{O}_2) .$$

We can use the connected sum formula in Theorem 6.6.1 to determine how $\mathfrak{L}$ changes under stabilizations of Legendrian links, see Section 3.5.

Proposition 6.6.4 *For every $L \hookrightarrow (M, \xi)$ as before we have that*

$$\mathfrak{L}(L^+, M, \xi) = U \cdot \mathfrak{L}(L, M, \xi) \quad \textit{and} \quad \mathfrak{L}(L^-, M, \xi) = \mathfrak{L}(L, M, \xi)$$

in $cHFL^-\left(-M, L, \mathfrak{t}_\xi + \mathrm{PD}[L]\right)$. Furthermore, it is

$$\widehat{\mathfrak{L}}(L^+, M, \xi) = [0] \quad \textit{and} \quad \widehat{\mathfrak{L}}(L^-, M, \xi) = \widehat{\mathfrak{L}}(L^-, M, \xi)$$

in $\widehat{HFL}(-M, L, \mathfrak{t}_\xi + \mathrm{PD}[L])$.

Proof. It follows from Theorem 6.6.1, which says that we just need to determine $\mathfrak{L}(\mathcal{O}^{\pm})$, together with the fact that $cHFK^{-}(\bigcirc) \cong \mathbb{F}[U]_{(0,0)}$, which says that $\mathfrak{L}(\mathcal{O}^{\pm})$ is fixed by the classical invariants of $\mathcal{O}^{\pm}$, and Equations (3.1) and (3.2) that give $\mathrm{tb}(\mathcal{O}^{\pm}) = -2$ and $\mathrm{rot}(\mathcal{O}^{\pm}) = \pm 1$.

The claim about $\widehat{\mathfrak{L}}$ is done in the same way using Corollary 6.6.3, but now we consider that $\widehat{HFL}(\bigcirc) \cong \mathbb{F}_{(0,0)}$. □

For the transverse case, we recall the following theorem from [30].

Theorem 6.6.5 (Epstein) *Two transverse links in a contact manifold are transversely isotopic if and only if they admit Legendrian approximations which differ by negative stabilizations.*

Hence, we can now prove that $\mathfrak{T}$ is a transverse invariant of links.

Proof of Theorem 6.2.5. It follows immediately from Proposition 6.6.4 and Theorem 6.6.5. □

Moreover, we have that the transverse invariant satisfies the following properties.

Corollary 6.6.6 *Take two transverse links T_1 and T_2 in contact manifolds (M_i, ξ_i) for $i = 1, 2$ as before. Then we have that the invariant of a connected sum is given by*

$$\mathfrak{T}(T_1 \# T_2, M_1 \# M_2, \xi_1 \# \xi_2) = \mathfrak{T}(T_1, M_1, \xi_1) \otimes \mathfrak{T}(T_2, M_2, \xi_2)\,,$$

while for the disjoint union we have that

$$\mathfrak{T}(T_1 \sqcup T_2, M, \xi) = \mathfrak{T}(T_1, M_1, \xi_1) \otimes \mathfrak{T}(T_2, M_2, \xi_2) \otimes \mathfrak{T}(T_{\mathcal{O}_2})\,,$$

where $T_{\mathcal{O}_2}$ denotes the transverse push-off of the Legendrian unknot $\mathcal{O}_2$.

Proof. It follows immediately from the definition of $\mathfrak{T}$, Theorem 6.6.1 and Proposition 6.6.2. □

6.6.2 Naturality and a refinement for $\mathfrak{L}$

The invariant $\widehat{\mathfrak{L}}(L, M, \xi)$ can be refined using a naturality property of the link Floer homology group of a connected sum. Suppose that L is a Legendrian link in a contact 3-sphere (S^3, ξ) such that $\widehat{\mathfrak{L}}(L, S^3, \xi) \neq [0]$. Let S be a convex, splitting sphere with connected dividing set, which intersects L transversely in exactly two points. Such a splitting sphere expresses L as a connected sum of two links L_1 and L_2.

Since $L = L_1 \#_S L_2$ then its hat Heegaard Floer homoloy group admits the splitting

$$\widehat{HFL}(L^*) \cong \widehat{HFL}\,(L_1^*) \otimes_{\mathbb{F}} \widehat{HFL}\,(L_2^*)\ ,$$

where the mirror images appear because S^3 has a diffeomorphism that reverses the orientation, then we identify $\widehat{HFL}(-S^3, L)$ with $\widehat{HFL}(S^3, L^*) := \widehat{HFL}(L^*)$.

The Alexander grading of $\widehat{\mathfrak{L}}(L, S^3, \xi)$ is well-defined, because we suppose that the invariant is non-zero. Moreover, we have that

$$A\left(\widehat{\mathfrak{L}}(L, S^3, \xi)\right) = A\left(\widehat{\mathfrak{L}}(L_1, S^3, \xi_1)\right) + A\left(\widehat{\mathfrak{L}}(L_2, S^3, \xi_2)\right)\ .$$

The pair (s_1, s_2), where

$$s_i = A\left(\widehat{\mathfrak{L}}(L_i, S^3, \xi_i)\right)$$

for $i = 1, 2$, is called *Alexander pair* of $\widehat{\mathfrak{L}}(L, S^3, \xi)$ respect to S and we denote it with $A_S(L, S^3, \xi)$. We have that the Alexander pair is an invariant of L in the sense of the following theorem.

Theorem 6.6.7 *Suppose that L is a Legendrian link in (S^3, ξ) such that $\widehat{\mathfrak{L}}(L, S^3, \xi)$ is non-zero. We also assume that there are two convex, splitting spheres S_1 and S_2, which decompose L as Legendrian connected sums, such that we can find a smooth isotopy of M that fix L and sends S_1 into S_2.*

Then the two Alexander pairs of $\widehat{\mathfrak{L}}(L, S^3, \xi)$, respect to S_1 and S_2, coincide, which means that $A_{S_1}(L, S^3, \xi) = A_{S_2}(L, S^3, \xi)$.

In particular, say $L = L_1 \#_{S_1} L'_1 = L_2 \#_{S_2} L'_2$, the hypothesis tells us that L_1 and L'_1 are smoothly isotopic to L_2 and L'_2 respectively.

Proof of Theorem 6.6.7. Suppose that D_1 and D_2 are Legendrian Heegaard diagrams for L_1 and L_2, while D'_1 and D'_2 are diagrams for L'_1 and L'_2 respectively. Moreover, denote with $D_1 \# D'_1$ and $D_2 \# D'_2$ the Legendrian Heegaard diagrams for L obtained from the corresponding Murasugi sums, as in Subsection 6.6.1.

In [57] it is proved that, given the map $\overline{\Phi}$ in the proof of Theorem 6.6.1, we can find a map $\overline{\Psi}$ such that the following diagram commutes.

$$\begin{tikzcd}
\widehat{HFL}(D_1) \otimes \widehat{HFL}(D'_1) \arrow[r, "\overline{\Phi}^*_{(D_1, D'_1)}"] \arrow[d, "\Psi^*_{(D_1,D_2)} \otimes \Psi^*_{(D'_1,D'_2)}"'] & \widehat{HFL}(D_1 \# D'_1) \arrow[d, "\overline{\Psi}"] \\
\widehat{HFL}(D_2) \otimes \widehat{HFL}(D'_2) \arrow[r, "\overline{\Phi}^*_{(D_2, D'_2)}"] & \widehat{HFL}(D_2 \# D'_2)
\end{tikzcd}$$

Moreover, it is shown that the map $\overline{\Psi}$ is an isomorphism which preserves the Alexander pair. Then this implies that, if the splitting spheres satisfiy the hypothesis of the theorem, the homology group $\widehat{HFL}(L^*)$ is canonically identified with $\widehat{HFL}(L_1^*) \otimes \widehat{HFL}(L_2^*)$ and the Legendrian invariant is given by

$$\widehat{\mathfrak{L}}(L, S^3, \xi) = \widehat{\mathfrak{L}}(L_1, S^3, \xi) \otimes \widehat{\mathfrak{L}}(L_2, S^3, \xi) \in \widehat{HFL}(L^*)\,.$$

This means precisely that the Alexander pair is well-defined and depends only on the isotopy class of S rel L. $\square$

We observe that this theorem is false if we consider the invariant $\mathfrak{L}$ instead. In fact, say K is the standard Legendrian unknot $\mathcal{O}$ after one positive stabilization, while denote with H the Legendrian knot $\mathcal{O}$ after two positive stabilizations. We immediately see that $K \# K$ is Legendrian isotopic to $\mathcal{O} \# H$, but the corresponding Alexander pairs are different

There is a version of Theorem 6.6.7 for transverse links.

Corollary 6.6.8 *Suppose that T is a transverse link in (S^3, ξ). Assume also that one of its Legendrian approximations L_T respects the hypothesis of Theorem 6.6.7. Then it is*

$$A_{S_1}(T, S^3, \xi) = A_{S_2}(T, S^3, \xi)\,,$$

where the Alexander pair is now defined as $A_S(T, S^3, \xi) = A_S(L_T, S^3, \xi)$.

Proof. It is a consequence of the fact that Legendrian approximations of the same transverse link differ by negative stabilizations. Therefore, the Alexander gradings are the same because negative stabilizations do not change the invariant, see Proposition 6.6.4. □

The Alexander pair can be useful in distinguishing Legendrian and transverse links that are not isotopic.

Proposition 6.6.9 *Suppose that L_1 and L_2 are smoothly isotopic Legendrian (transverse) links in (S^3, ξ) which appear as in one of these two cases:*

1. *say $L_1 \approx L_2 = K \#_S H$, where K and H are Legendrian (transverse) knots with prime knot types;*

2. *say $L_1 \approx L_2$ is a 2-component Legendrian (transverse) link, obtained from three Legendrian (transverse) knots K, H and J with prime knot types, defined as follows: take the connected sum of $K \#_S H$ with a (standard) positive Hopf link $\mathcal{H}_+$ and J, in the way that $K \#_S H$ is summed on the first component of $\mathcal{H}_+$ and J on the second one.*

We have that if $A_S(L_1, S^3, \xi) \neq A_S(L_2, S^3, \xi)$ then L_1 is not Legendrian (transverse) isotopic to L_2.

Proof. It follows from Theorem 6.6.7 and Corollary 6.6.8 and the fact that, if there is a Legendrian (transverse) isotopy F between L_1 and L_2, the isotopy F is such that $F(S) = S'$ and, in both cases, we can smoothly isotope S' onto S. □

Chapter 7

Loose and non-loose links

7.1 Legendrian links in overtwisted structures

Legendrian links in overtwisted contact 3-manifolds come in two types: loose and non-loose. A Legendrian link is *loose* if also its complement is overtwisted, while it is *non-loose* if the complement is tight. More explicitely, a Legendrian link is loose if and only if we can find an overtwisted disk that is disjoint from the link.

We observe that if L_1 is loose then $L_1 \# L_2$ is also loose. This can be stated in an equivalent way by saying that if a Legendrian link $L_1 \# L_2$ is non-loose then both L_1 and L_2 are non-loose or live in a tight contact manifold.

On the other hand, the connected sum of two non-loose Legendrian links can be loose. In fact, denote with ξ_i the overtwisted structure on S^3 with Hopf invariant equals to $i \in \mathbb{Z}$, where ξ_0 is homotopic as a plane field to the unique tight structure on S^3, see Subsection 3.1.1. From [24] we have that in (S^3, ξ_{-1}) there is a non-loose Legendrian unknot K. Then the knot $K \# K$ is a loose Legendrian unknot in (S^3, ξ_{-2}), because in [24] it is also proved that non-loose unknots in S^3 only appear in the structure ξ_{-1}.

Since $\widehat{c}(M, \xi)$ is always zero for overtwisted contact manifold [70], we have that the Legendrian link invariant $\mathfrak{L}$ is always torsion in this case. But more can be said for loose links.

Proposition 7.1.1 *Let L be a loose Legendrian link in an overtwisted contact 3-manifold (M, ξ). Then we have that $\mathfrak{L}(L, M, \xi) = [0]$.*

Proof. The complement of L in M contains an overtwisted disk E. Since E is contractible, we can find a ball U such that $E \subset U \subset M \setminus L$. The restriction of ξ to U is obviously overtwisted, moreover we can choose E such that ∂U has trivial dividing set. Thus we have that $(M, \xi) = (M, \xi_1) \# (S^3, \xi')$; where ξ_1 concides with ξ near L and ξ' is an overtwisted structure on S^3.

We now use the fact that the standard Legendrian unknot $\mathcal{O}$ is well-defined, up to Legendrian isotopy, and Theorem 6.6.1 to say that

$$\mathfrak{L}(L, M, \xi) = \mathfrak{L}(L_1, M, \xi_1) \otimes \mathfrak{L}(\mathcal{O}, S^3, \xi') \, .$$

Since $cHFK^-(\bigcirc) \cong \mathbb{F}[U]_{(0,0)}$, and then there is no torsion, we obtain that the Legendrian invariant of an unknot is zero in overtwisted 3-spheres. Then $\mathfrak{L}(L, M, \xi)$ is also zero. $\square$

We can use this proposition to say something about stabilizations. In fact, in principle a stabilization of a non-loose Legendrian link $L \hookrightarrow (M, \xi)$ can be loose, but if $\mathfrak{L}(L, M, \xi)$ is non zero then all its negative stabilizations are also non-loose.

It is easy to prove that we can always find loose Legendrian links in every overtwisted contact 3-manifold; in fact Legendrian links inside a Darboux ball need to be loose, see Proposition 3.5.1, and we have Darboux balls centered at every point. On the other hand, it is a little harder to show that the same holds for non-loose Legendrian links.

Proposition 7.1.2 *In every (M, ξ) with ξ overtwisted there exists a non-loose, non-split n-component Legendrian link such that all the components are also non-loose.*

Proof. It is a known theorem in contact topology that every overtwisted contact 3-manifold is obtained from some -1-surgeries and exactly one $+1$-surgery on Legendrian knots in (S^3, ξ_{st}), see [27].

Take a surgery presentation for (M, ξ) and denote by $J = J^+ \cup J^-$ the corresponding framed Legendrian link in (S^3, ξ_{st}); which can be represented by a front projection. The component J^+ is the knot where we perform the $+1$-surgery. We claim that the Legendrian

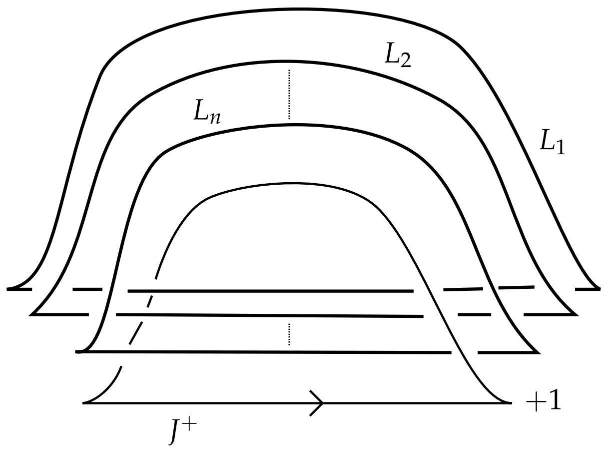


Figure 7.1: The knots $L_1, L_2, ..., L_n$ are the components of L.

link L, given by n parallel contact push-offs of J^+ as shown in Figure 7.1, is what we wanted. Note that the contact push-off of a Legendrian knot K is a knot K', which is parallel to K except for a small tangle where it twists along it; the number of twists is equal to $\mathrm{tb}(K)$.

We start by proving that each L_i is non-loose; this implies that also L is non-loose. Suppose that there is an overtwisted disk E in $M \setminus L_i$. Thus E will remain unchanged if we apply a -1-surgery along L_i and then the resulting contact manifold (M^i, ξ^i) is overtwisted. But it is easy to see that two surgeries, with opposite signs, along contact push-offs cancel each other and this means that the new manifold has a surgery presentation given by only -1-surgeries. Now from Theorem 3.1.9 we obtain that (M^i, ξ^i) is Stein fillable and then from [70] we know that it has to be tight. This is a contradiction.

We only have to show that L is non-split. If not we can find a connected sum decomposition $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$ and say L_1 is embedded in M_1 and L_2 in M_2. Since we can suppose that (M_1, ξ_1) is overtwisted we have that L_2 is loose in (M, ξ), but this is impossible because we already proved before that it is non-loose. $\square$

Now take a Darboux ball B in our overtwisted (M, ξ). We saw in Proposition 3.5.1 that we can always find an overtwisted disk in M, which is disjoint from B. We can deform such an overtwisted disk to intersect B and, since we do not touch the boundary, it remains an overtwisted disk. This means that every Darboux ball, in an overtwisted contact 3-manifold, intersects at least one overtwisted disk. More generally, the same argument tells us that if $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$ then we always find overtwisted disks that are not entirely contained in one of the two summands of (M, ξ).

The previous observation suggests that the connected sum of a non-loose Legendrian link with another Legendrian link, which is non-loose or lives in a tight structure, may still be loose. On the other hand, the following proposition holds for disjoint unions.

Proposition 7.1.3 *The disoint union of a non-loose Legendrian link $L_1 \hookrightarrow (M_1, \xi_1)$ with another Legendrian link $L_2 \hookrightarrow (M_2, \xi_2)$, which is non-loose or lives in a tight structure is also non-loose.*

Furthermore, if L is non-loose then L is a split Legendrian link if and only if its smooth link type is split.

Proof. For the first statement we observe that ξ_i restricted to $M_i \setminus L_i$ is tight for $i = 1, 2$. Then $\xi = \xi_1 \# \xi_2$ restricted to $M \setminus L = M_1 \# M_2 \setminus (L_1 \sqcup L_2)$ is also tight. The second statement follows from Colin's result in [13]. $\square$

7.2 A classification theorem for loose knots

While it was known that non-loose Legendrian knots are not classified by their classical invariants [57], in the case of loose knots such example was found only recently by Vogel [89]; even though, according to [24], this phenomenon was already known to Chekanov. Conversely, there were some results that go in the opposite direction.

Etnyre's coarse classification of loose Legendrian knots [29] is probably the most important one. It says that loose knots are completely determined by their classical invariants, but only up to contactomorphism. Another result was proved by Dymara in [17] and it states that two Legendrian knots, with the same classical invariants, such that the complement of their union contains an overtwisted disk are Legendrian isotopic.

Theorem 7.2.1 (Dymara) *Consider a rational homology overtwisted 3-sphere (M, ξ). Then two loose Legendrian knots K_1 and K_2 in (M, ξ) with the same classical invariants and such that $K_1 \cup K_2$ is loose are Legendrian isotopic.*

In this thesis we show that Dymara's result can be strengthened. In fact we prove the following theorem.

Theorem 7.2.2 *Consider a rational homology contact 3-sphere (M, ξ). Suppose that there are two loose Legendrian knots K_1 and K_2 in (M, ξ) and denote with (E_1, E_2) a pair of overtwisted disks, where E_i is contained in the complement of K_i for $i = 1, 2$. If we can find E_1 and E_2 disjoint then K_1 and K_2 are Legendrian isotopic if and only if they have the same classical invariants.*

Though we still need an assumption on the overtwisted disks, this version can be applied in many interesting cases like disjoint unions of Legendrian knots.

The proof of Theorem 7.2.2 only requires to show that loose Legendrian knots with the same classical invariants are Legendrian isotopic.

Proof of Theorem 7.2.2. The idea of the proof is to find another Legendrian knot K, with the same classical invariants as K_1 and K_2, and to show that there are overtwisted disks in the complement of both $K_1 \cup K$ and $K_2 \cup K$. Then Theorem 7.2.1 gives that K_1 and K_2 are both Legendrian isotopic to K.

If there is an overtwisted disk in the complement of $K_1 \cup K_2$ then the claim follows immediately from Theorem 7.2.1. Then we suppose that this is not the case.

Since E_1 and E_2 are disjoint disks we can find two closed balls B_1 and B_2 such that $E_i \subset B_i$ for $i = 1, 2$. Moreover, we can suppose that each B_i is disjoint from K_i; this is because we start from the assumption that each E_i is in the complement of K_i.

Now we have that the contact manifold $(B_i, \xi|_{B_i})$ is an overtwisted D^3. Then from Eliashberg's classification of overtwisted structures [18, 22] we know that $(B_i, \xi|_{B_i})\#(S^3, \xi_0)$ is contact isotopic to $(B_i, \xi|_{B_i})$. This holds since the contact connected sum with (S^3, ξ_0)

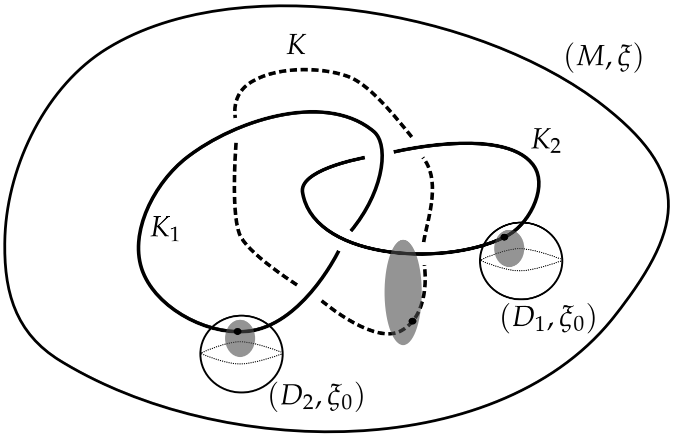


Figure 7.2: Three overtwisted disks are drawn in grey.

does not change the homotopy class of the 2-plane field given by ξ. From this we have that inside $(B_i, \xi|_{B_i})$ there is a sphere with trivial dividing set which bounds a closed ball D_i such that $\xi|_{D_i}$ is contact isotopic to ξ_0. Moreover, both $\mathring{D}_i$ and $B_i \setminus D_i$ contains some overtwisted disks from Proposition 3.5.1.

At this point, we have found two closed balls D_1 and D_2 in (M, ξ) such that each D_i is in the complement of K_i and $\xi|_{D_i}$ is contact isotopic to the overtwisted structure ξ_0; furthermore, the contact manifold $M \setminus (D_1 \cup D_2)$ is also overtwisted. This situation is pictured in Figure 7.2. Let us denote with M' the overtwisted manifold $M \setminus D_2$; the boundary of D_1 gives a contact connected sum decomposition of M' where the components are $M \setminus (D_1 \cup D_2)$ and D_1. Then the same argument that we applied before tells us that $(M \setminus (D_1 \cup D_2), \xi)$ is contact isotopic to (M', ξ) without a Darboux ball.

Since K_2 is a Legendrian knot in M' and we can suppose that a Darboux ball is missed by it, we can identify K_2 with a Legendrian knot K inside $M \setminus (D_1 \cup D_2)$ with the same classical invariants as K_2, which are assumed to coincide with the ones of K_1. The Legendrian knot K is disjoint from both the overtwisted balls D_i and then we find overtwisted disks in the complement of both $K_i \cup K$. This concludes the proof. $\square$

The condition on the overtwisted disks in Theorem 7.2.2 cannot be removed. Vogel in [89] gives an example of two loose Legendrian unknots, both with Thurston-Bennequin number equal to zero and rotation number equal to one, that are not Legendrian isotopic.

In fact, although we can find overtwisted disks in the complement of both knots such disks always intersect each other.

An important case where the situation described in Theorem 7.2.2 appears is when we have a disjoint union of two Legendrian knots.

From Theorem 3.5.2 we observe that if ξ is tight then L is a split Legendrian link if and only if its smooth link type is split.

Corollary 7.2.3 *Suppose K_1 and K_2 are two loose Legendrian knots in the rational homology contact sphere (M, ξ) such that $K_1 \cup K_2$ is a split Legendrian link. Then they are Legendrian isotopic if and only if they have the same classical invariants.*

Proof. We have that $(M, \xi) = (M_1 \# M_2, \xi_1 \# \xi_2)$ and $K_i \hookrightarrow (M_i, \xi_i)$. Hence, if K_1 is loose in (M_1, ξ_1) then clearly we can apply Theorem 7.2.1; on the other hand, if (M_1, ξ_1) is tight then Theorem 3.5.2 gives that K_2 is loose in (M_2, ξ_2) and then we can use the same argument. We only have to consider the case when both K_i are non-loose in (M_i, ξ_i). Then we apply Theorem 7.2.2, since we can find overtwisted disks in both summands using Proposition 3.5.1. □

We conclude with the following observation. Let us consider two loose Legendrian knots K_1 and K_2 in (M, ξ) with same classical invariants and such that $L = K_1 \cup K_2$ is a topologically split 2-component Legendrian link. Corollary 7.2.3 says that if K_1 is not Legendrian isotopic to K_2 then L is non-split as a Legendrian link. The example of Vogel that was mentioned before falls into this case.

7.3 Existence of non-loose links with loose components

In Section 7.1 we found non-loose Legendrian links with non-loose components. A more interesting result is to show that, under some hypothesis on (M, ξ), we can also find non-split n-component Legendrian and transverse links such that $\mathfrak{L}(L, M, \xi) \neq [0]$, which means that they are non-loose from Proposition 7.1.1, and all of their components are instead loose. We start by constructing Legendrian knots with non trivial invariant in all the overtwisted structures on S^3.

Consider the family of Legendrian knots $L(j)$, where $j \geqslant 1$, given by the surgery diagram in Figure 7.3. Using Kirby calculus we easily see that $L(j)$ is a positive torus knot $T_{2,2j+1}$ in S^3. On the other hand, if we want to determine the Legendrian invariants of $L(j)$ and the contact structure where it lives we need the following proposition, whose proof is found in [57].

Let us introduce the notation. Suppose that the link $J = J^+ \cup J^- \subset (S^3, \xi_{st})$ is a contact (± 1)-surgery presentation of a contact 3-manifold (M, ξ) and L is a Legendrian knot in (S^3, ξ_{st}), disjoint from J. Hence, L represents a Legendrian knot also in (M, ξ). Let tb_0 and rot_0 denote the Thurston-Bennequin and rotation number of L as a knot in the standard S^3. Writing $J = J_1 \cup ... \cup J_t$, let a_i be the integral surgery coefficient on the link component J_i; i.e. $a_i = \mathrm{tb}(J_i) \pm 1$ if $J_i \in J^\pm$. Define the linking matrix

$$Q(a_0, a_1, ..., a_t) = (q_{i,j})_{i,j=0}^t ;$$

where

$$q_{i,j} = \begin{cases} a_i & \text{if } i = j \\ \mathrm{lk}(J_i, J_j) & \text{if } i \neq j \, , \end{cases}$$

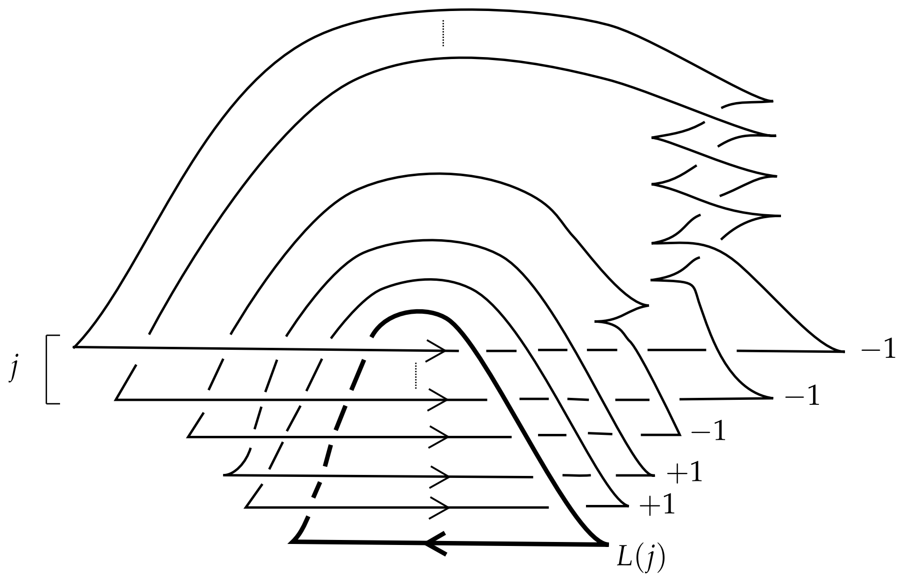


Figure 7.3: Contact surgery presentation for the Legendrian knot $L(j)$.

with the convention that $L = J_0$ and $a_0 = 0$. Similarly, let $Q = Q(a_1, ..., a_t)$ denote the matrix $(q_{i,j})_{i,j=1}^{t}$.

Proposition 7.3.1 *Suppose that J is a surgery presentation of a Legendrian knot L in the contact manifold (M, ξ), as before. Then the Hopf invariant of ξ is determined by the following equation:*

$$d_3(\xi) = \frac{1}{4}\left[\langle \begin{pmatrix} rot(J_1) \\ \vdots \\ rot(J_t) \end{pmatrix}, Q^{-1} \begin{pmatrix} rot(J_1) \\ \vdots \\ rot(J_t) \end{pmatrix} \rangle - 3\sigma(Q) - 2t \right] + \left|J^+\right| .$$

Furthermore, the Thurston-Bennequin and rotation numbers of L can be extracted by the formulae:

$$tb(L) = tb_0 + \frac{det(Q(0, a_1, ..., a_t))}{det(Q)}$$

and

$$rot(L) = rot_0 - \langle \begin{pmatrix} rot(J_1) \\ \vdots \\ rot(J_t) \end{pmatrix}, Q^{-1} \begin{pmatrix} lk(L, J_1) \\ \vdots \\ lk(L, J_t) \end{pmatrix} \rangle .$$

We apply this proposition to the knots in Figure 7.3 and we immediately obtain that they are Legendrian knots in (S^3, ξ_{1-2j}) and their invariants are:

- $\text{tb}(L(j)) = 6 + 4(j-1)$;
- $\text{rot}(L(j)) = 7 + 6(j-1)$.

Moreover, from [57] we also know that $\widehat{\mathfrak{L}}(L(j), S^3, \xi_{1-2j}) \neq [0]$ and then $\mathfrak{L}(L(j), S^3, \xi_{1-2j})$ is a non-zero torsion class in $HFK^-(T_{2,-2j-1})$. Both have bigrading $(1, 1-j)$.

Now we want to consider another family of Legendrian knots: the knots $L_{k,l}$, with $k, l \geqslant 0$, shown in Figure 7.4. From [57] we also know that $L_{k,l}$ is a negative torus knot $T_{2,-2k-2l-3}$ in (S^3, ξ_{2l+2}) and its invariants are:

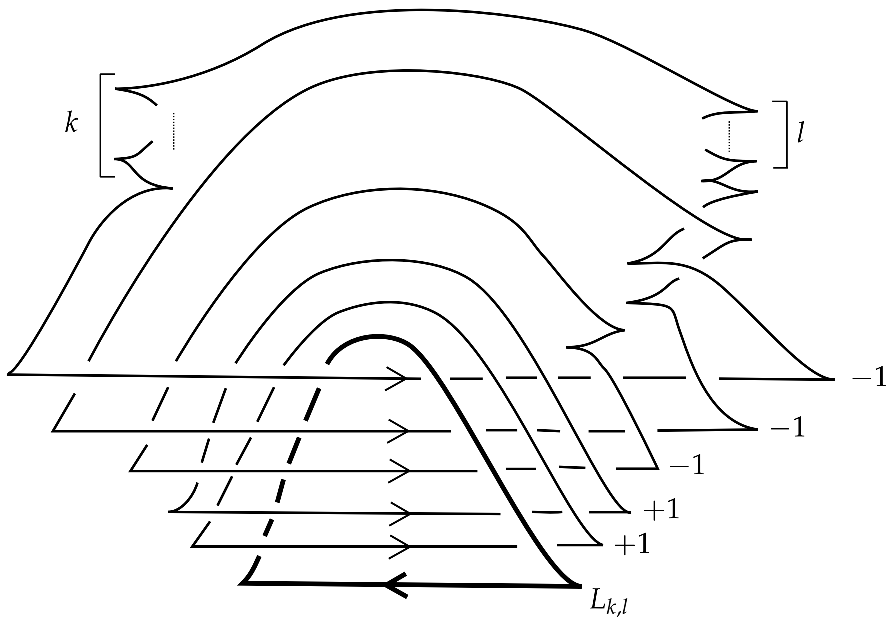


Figure 7.4: Contact surgery presentation for the Legendrian knot $L_{k,l}$.

- $\text{tb}(L_{k,l}) = -6 - 4(k + l)$;
- $\text{rot}(L_{k,l}) = -7 - 2k - 6l$.

In this case, from [57], we have that the invariant $\widehat{\mathfrak{L}}$ of the Legendrian knots $L_{0,l}$, $L_{1,1}$ and $L_{1,2}$ is non-zero with bigrading $(-2k, 1 - k + l)$ in the homology group $\widehat{HFK}(T_{2,2k+2l+3})$. Obviously, the fact that $\widehat{\mathfrak{L}}$ is non-zero again implies that the same is true for the invariant $\mathfrak{L}$.

At this point, we define the Legendrian knots K_i, for every $i \in \mathbb{Z}$, in the following way:

$$K_i = \begin{cases} L(j)\#L(1) & \text{if } i = -2j < 0 \\ L(j) & \text{if } i = 1 - 2j < 0 \\ L_{0,j-1} & \text{if } i = 2j > 0 \\ L_{0,j-1}\#L(1) & \text{if } i = 2j - 1 > 0 \\ L_{0,0}\#L(1)^2 & \text{if } i = 0\,. \end{cases}$$

Then we have the following result.

Proposition 7.3.2 (Lisca, Ozsváth, Stipsicz and Szabó) *The Legendrian knot $K_i \hookrightarrow (S^3, \xi_i)$ is such that $\mathfrak{L}(K_i, S^3, \xi_i) \neq [0]$ and then it is non-loose for every $i \in \mathbb{Z}$.*

Proof. It follows easily from the previous computation and the connected sum formula. $\square$

We can now go back to links. Let us take an overtwisted 3-manifold (M, ξ) such that there exists another contact structure ζ on M with $\widehat{c}(M, \zeta) \neq [0]$ and $\mathfrak{t}_\xi = \mathfrak{t}_\zeta$; in particular ζ is tight. Consider $\mathcal{O}$ the standard Legendrian unknot in (M, ζ). We have that $\mathfrak{L}(\mathcal{O}, M, \zeta)$ coincides with $\mathbf{e}_{-d_3(M,\zeta),0} \neq [0]$ and the invariant is non-torsion; this is because $HFK^-\left(-M, \mathcal{O}, \mathfrak{t}_\zeta\right) \cong \mathbb{F}[U]_{(-d_3(M,\zeta),0)}$ and Theorem 6.5.3.

Fix B a Darboux ball in (M,ζ), which contains $\mathcal{O}$. Since B is contact isotopic to $(\mathbb{R}^3,\xi_{\mathrm{st}})$, we can take the Legendrian link H_+^{n-1} in B, that is defined as $n-1$ connected sums of the standard positive Legendrian Hopf link; see Figure 7.5.

If $n = 1$ we do not do anything and we keep the the knot $\mathcal{O}$. The invariant

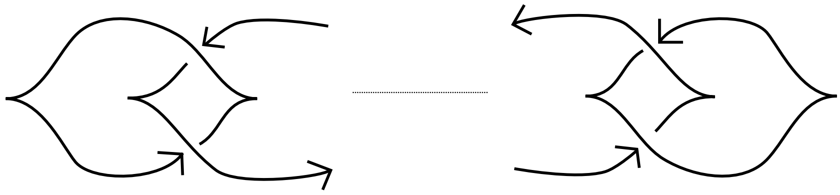


Figure 7.5: Connected sums of $n-1$ standard Legendrian positive Hopf links.

$\mathfrak{L}(H_+^{n-1},M,\zeta)$ is the tensor product of $\mathfrak{L}(\mathcal{O},M,\zeta)$, which as we said is non-zero, with $n-1$ times $\mathfrak{L}(H_+)$ the invariant in the standard S^3. An easy computation, see also Subsection 1.5.2, shows that $\mathfrak{L}(H_+)$ is the only non-torsion element in the group

$$cHFL^-(H_-)\cong\mathbb{F}[U]_{(0,0)}\oplus\mathbb{F}[U]_{(1,1)}\oplus\left(\frac{\mathbb{F}[U]}{U\cdot\mathbb{F}[U]}\right)_{(0,0)}$$

with bigrading $(1,1)$. This means that not only $\mathfrak{L}(H_+^{n-1},M,\zeta)$ is non-torsion, but it is also represented by the top generator of one of the $\mathbb{F}[U]$ towers of $cHFL^-\left(-M,H_+^{n-1},\mathfrak{t}_\zeta\right)$. Here the negative Hopf link H_- appears because it is the mirror image of H_+ and the Legendrian invariant lives in its homology group, as from the definition in Chapter 6.

Now we perform a connected sum between H_+^{n-1} and the Legendrian knot $L(1)$, defined before; where we suppose that $L(1)$ is summed to H_+^{n-1} on the rightmost component in Figure 7.5. We saw that $L(1)$ is a positive trefoil in (S^3,ξ_{-1}). Thus $H_+^{n-1}\# L(1)$ lives in (M,ξ'); where ξ' is an overtwisted structure such that $\mathfrak{t}_{\xi'}=\mathfrak{t}_\zeta=\mathfrak{t}_\xi$ and $d_3(\xi')=d_3(\zeta)-1$.

If we do another connected sum with the Legendrian knot $K_{d_3(\xi)-d_3(\zeta)+1}$, this time on the leftmost component of H_+^{n-1}, we obtain the Legendrian link

$$L=K_{d_3(\xi)-d_3(\zeta)+1}\# H_+^{n-1}\# L(1)\,;$$

which is a link in M equipped with a contact structure that has the same Hopf invariant as (M,ξ) and induces the same Spin^c structure of ξ. From Eliashberg's classification of overtwisted structures [18], we conclude that L is a Legendrian n-component link in (M,ξ). We can now prove the following theorem.

Theorem 7.3.3 *In every overtwisted 3-manifold (M,ξ), such that there exists a contact structure ζ with $\mathfrak{t}_\xi=\mathfrak{t}_\zeta$ and $\widehat{c}(M,\zeta)\neq[0]$, there is a non-split n-component Legendrian link L, for every $n\geqslant 1$, such that $\mathfrak{L}(L,M,\xi)$ is non-zero and all of its components are loose. In particular, L is non-loose and stays non-loose after a sequence of negative stabilizations.*

Furthermore, the transverse link T, obtained as transverse push-off of L, has exactly the same properties of L.

Proof. We already saw that the link L exists if the hypothesis of the theorem holds. So first we check that $\mathfrak{L}(L,M,\xi)$ is non-zero. In fact, the invariant is represented by the tensor product of a non-zero torsion element with $\mathfrak{L}(H_+^{n-1},M,\zeta)$. We are working with $\mathbb{F}[U]$-modules and we recall, see Subsection 1.5.2, that

$$\mathbb{F}[U]\otimes_{\mathbb{F}[U]}\left(\frac{\mathbb{F}[U]}{U\cdot\mathbb{F}[U]}\right)\cong\frac{\mathbb{F}[U]}{U\cdot\mathbb{F}[U]}$$

and, more precisely, in the $\mathbb{F}[U]$ factor only the generator survives. Then, from what we said before, we have that $\mathfrak{L}(L, M, \xi)$ remains non-zero. Note that this is false if instead we take the negative Hopf link. In this case the invariant $\mathfrak{L}$ does not lie in the top of an $\mathbb{F}[U]$ tower of the homology group and then it vanishes after the tensor product.

This immediately implies that $\mathfrak{T}(T, M, \xi)$ is also non-zero and then the theorem holds for transverse links. Moreover, the fact that the components of L (and T) are loose follows easily from the construction of L. In fact, we performed connected sums of (M, ζ) with two overtwisted S^3's and each component lies in one of these three summands. It is only left to prove that L is non-split.

From [13] we know that the connected sum of two tight contact manifolds is still tight. This implies that a non-loose Legendrian link is split if and only if its smooth link type is split. Hence, we just have to show that L is non split as a smooth n-component link. But L is a connected sum of torus links in a 3-ball inside M and we know that L is non-split as a link in S^3. Furthermore, if L is split in M then it would be split also in the 3-sphere and this is a contradiction. □

7.4 Non-simple link types

If two Legendrian links L_1 and L_2 are Legendrian isotopic then they have contactomorphic complements. This means that if L_1 is loose (we suppose that the L_i's are in an overtwisted contact manifold) and L_2 is non-loose then they are not Legendrian isotopic.

We recall that the link type $\mathcal{L}(L)$ in M, which is the smooth isotopy class of the Legendrian link L in M, is called *Legendrian simple* if $\mathcal{L}(L_1) = \mathcal{L}(L_2)$ and L_1, L_2 have the same classical Legendrian invariants implies that L_1 is Legendrian isotopic to L_2. This means that finding a pair of Legendrian knots in (M, ξ) overtwisted, one loose and the other non-loose, but with the same classical invariants (knot type, Thurston-Bennequin and rotation number) proves that they do not have Legendrian simple knot type.

From the work of Eliashberg and Fraser [24] and Etnyre [29] we know that such pairs really exist.

Proposition 7.4.1 *In (S^3, ξ_1) the unknot is not a Legendrian simple knot type; where ξ_1 is the overtwisted structure on the 3-sphere with $d_3(\xi_1) = 1$.*

In fact, as we said in Section 7.1, in (S^3, ξ_1) there is a non-loose Legendrian unknot K with $\mathrm{tb}(K) = 1$ and $\mathrm{rot}(K) = 0$, while from [29] we also have that, in the same contact manifold, there is a loose Legendrian unknot K' with same classical invariants.

It is harder to find two non-loose Legendrian links L_1 and L_2 with same classical invariants and such that they are not Legendrian isotopic. The first example of such links was given by Etnyre in [28] and others were found by Lisca, Ozsváth, Stipsicz and Szabó in [57]; from which we report the following result.

Proposition 7.4.2 (Lisca, Ozsváth, Stipsicz and Szabó) *The knot type $T_{(2,-7)} \# T_{(2,-9)}$ in (S^3, ξ_{12}) contains two non-loose Legendrian knots, with equal Thurston-Bennequin and rotation numbers, which are not Legendrian isotopic.*

The Legendrian isotopy classes of the knots in this proposition are distinguished using the refinement of the Legendrian invariant $\widehat{\mathfrak{L}}$ for knots.

In the previous subsection we saw that, under some hypothesis, in an overtwisted 3-manifold (M,ξ) we can find non-loose, non-split n-component Legendrian links L_n. Consider the links L'_n obtained as the connected sum of L_n with the standard Legendrian unknot in (S^3,ξ_0), where ξ_0 is the overtwisted S^3 with zero Hopf invariant.

Since (M,ξ) is already overtwisted, we have that $(M,\xi)\#(S^3,\xi_0)$ is contact isotopic to (M,ξ). This means that L'_n is also a non-split Legendrian link in (M,ξ), which is smoothly isotopic to L_n for every $n \geqslant 1$, but unlike L_n it is clearly loose.

Each component of L'_n has the same classical invariants of a component of L_n. Moreover, if $n \geqslant 2$ then there is an overtwisted disk in their complement. From Dymara's Theorem 7.2.1, the components of L'_n are Legendrian isotopic to the ones of L_n. Hence, we have the following corollary.

Corollary 7.4.3 *The link type of L_n and L'_n in M, which is denoted with $\mathcal{L}$, is both Legendrian and transverse non-simple.*

We can also find non-simple link types where the two Legendrian and transverse representatives are non-loose.

Proposition 7.4.4 *Let us consider the links $L_1 = (L_{0,2}\# L_{1,2})\# H_+\# L(1)$ and $L_2 = (L_{1,1}\# L_{0,3})\# H_+\# L(1)$ in the contact manifold (S^3,ξ_{11}); where in L_i the knots on the left are summed on the first component of H_+ and $L(1)$ on the second one. Then L_1 and L_2 are two non-loose, non-split 2-component Legendrian links, with the same classical invariants and Legendrian isotopic components, but that are not Legendrian isotopic.*

In the same way, the transverse push offs of L_1 and L_2 are two non-loose, non-split 2-component transverse links, with the same classical invariants and transversely isotopic components, but that are not transversely isotopic.

Proof. We apply Theorem 6.6.7. The Legendrian invariant of L_1 and L_2 is computed in [57] and it is non-zero; moreover, the Alexander pairs of $\widehat{\mathfrak{L}}(L_1,S^3,\xi_{11})$ and $\widehat{\mathfrak{L}}(L_2,S^3,\xi_{11})$, respect to splitting spheres separating $L_{0,2}$ from $L_{1,2}$ and $L_{1,1}$ from $L_{0,3}$, are different. The fact that the components are Legendrian isotopic follows from Theorem 7.2.1. The same argument proves the theorem in the transverse setting. □

Using the same construction, the refined version of $\widehat{\mathfrak{L}}$ and $\widehat{\mathfrak{T}}$ can be applied to find such examples for links with more than two components in every contact manifold as in Theorem 7.3.3.

Theorem 7.4.5 *Suppose that (M,ξ) is an overtwisted 3-manifold as in Theorem 7.3.3. Then in (M,ξ) there is a pair of non-loose, non-split n-component Legendrian (transverse) links, with the same classical invariants and Legendrian (transversely) isotopic components, but that are not Legendrian (transversely) isotopic.*

Proof. Let us take a standard Legendrian (transverse) positive Hopf link H_+ in (S^3,ξ_{st}). On the first component of H_+ we perform a connected sum with the knot $L_{0,2}\# L_{1,2}$ in one case and $L_{1,1}\# L_{0,3}$ in the other. While, on the second component of H_+, we sum a non-split Legendrian (transverse) n-component link in the overtwisted manifold (M,ξ'), where $d_3(\xi') = d_3(\xi) - 12$, with non-zero invariants; those links exist as we know from Theorem 7.3.3. We conclude by applying the same argument of the proof of Proposition 7.4.4. □

Chapter 8

Link Floer homology and grid diagrams

8.1 Filtered simply blocked link grid homology

8.1.1 The chain complex

Grid diagrams are simple combinatorial presentation of links in S^3, dating back to the 19th century. A grid diagram is an $l \times l$ grid of squares, l of which are marked with an O and l of which are marked with an X. Each row and column contains exactly one O and one X. A projection of a link together with an orientation on it can be associated to a grid diagram D. These grids can be used to give a simpler reformulation of link Floer homology, called *grid homology*. Of course these two homologies are isomorphic, nevertheless grid homology can be easier to study.

In this thesis we use the same notation of the book "Grid homology for knots and links" [67]. In this book particular attention is given to two versions of the grid homology of a link L: the *simply blocked grid homology* $\widehat{GH}(L)$ and the *collapsed unblocked grid homology* $cGH^-(L)$; both these homology groups are invariant under link equivalence. We study a slightly different version of $\widehat{GH}(L)$. Let us denote with $\mathbb{F}$ the field with two elements; we start constructing a filtered $\mathbb{F}$-complex $\left(\widehat{GC}(D), \widehat{\partial}\right)$ from a grid diagram D, equipped with an increasing $\mathbb{Z}$-filtration $\mathcal{F}$ and we prove that $\mathcal{F}$ induces a filtration in homology, leading to the filtered homology group $\widehat{\mathcal{GH}}(L)$. The latter is not completely unrelated to $\widehat{GH}(L)$ as we see later in this section.

We always suppose that a link is oriented. We denote by D a toroidal grid diagram that represents an n-component link L. The number $\text{grd}(D)$ is the number of rows and columns in the grid. The orientation in the diagram is taken by going from the X to the O-markings in the columns and the opposite in the rows. Vertical lines are numbered from left to right and horizontal lines from bottom to top, as shown in Figure 8.1. We identify the boundaries of the grid in order to make it a fundamental domain of a torus; then the lines of the diagram are embedded circles in this torus. Any $\text{grd}(D)$-tuple of points x in the grid, with the property that each horizontal and vertical circle contains exactly one of the elements of x, is called a *grid state* of D.

Consider the set of the O-markings $\mathbb{O} = \{O_1, ..., O_{\text{grd}(D)}\}$. We call *special O-markings* a non-empty subset $s\mathbb{O} \subset \mathbb{O}$ that contains at most one O-marking from each component of L, while we call the others normal O-markings. We represent the special ones with a double

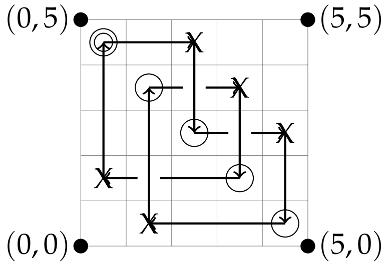


Figure 8.1: A grid diagram of the positive trefoil knot.

circle in the grid diagram. We usually consider only the case when $|s\mathbb{O}| = n$, which means there is exactly one special O-marking on each component. We talk about the general case in Subsection 8.2.2. From now on there are always n special O-markings in a grid diagram, unless it is explicitly written differently.

We define the *simply blocked complex* $\widehat{GC}(D)$ as the free $\mathbb{F}[V_1, ..., V_{\mathrm{grd}(D)-n}]$-module, where $\mathbb{F} = \mathbb{Z} \setminus 2\mathbb{Z}$, over the grid states $S(D) = \{x_1, ..., x_{\mathrm{grd}(D)!}\}$.

We associate to every grid state x the integer $M(x)$, called the *Maslov grading* of x, defined as follows:

$$M(x) = M_{\mathbb{O}}(x) = \mathcal{J}(x - \mathbb{O}, x - \mathbb{O}) + 1\,; \tag{8.1}$$

where

$$\mathcal{J}(P, Q) = \sum_{a \in P} \Big| \{(a, b) \in (P, Q) \mid b \text{ has both coordinates strictly bigger than the ones of } a\} \Big|$$

with coordinates taken in the interval $[0, \mathrm{grd}(D))$.

Then we have the Maslov $\mathbb{F}$-splitting

$$\widehat{GC}(D) = \bigoplus_{d \in \mathbb{Z}} \widehat{GC}_d(D)$$

where $\widehat{GC}_d(D)$ is the finite dimensional $\mathbb{F}$-vector space generated by the elements $V_1^{l_1} \cdot ... \cdot V_m^{l_m} x$, with $x \in S(D)$ and $m = \mathrm{grd}(D) - n$, such that

$$M(V_1^{l_1} \cdot ... \cdot V_m^{l_m} x) = M(x) - 2 \sum_{i=1}^{m} l_i = d\,.$$

We define another integer-valued function on grid states, the *Alexander grading* $A(x)$, with the formula

$$A(x) = \frac{M(x) - M_{\mathbb{X}}(x)}{2} - \frac{\mathrm{grd}(D) - n}{2}\,;$$

where $M_{\mathbb{X}}(x)$ is defined in Equation (8.1), replacing the set $\mathbb{O}$ with $\mathbb{X}$. For the proof that $A(x)$ is an integer we refer to [67].

Now we introduce an increasing filtration on $\widehat{GC}(D)$, see Subsection 1.4.1 such that

$$\mathcal{F}^s \widehat{GC}(D) = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^s \widehat{GC}_d(D)$$

and where $\mathcal{F}^s\widehat{GC}_d(D)$ is generated over $\mathbb{F}$ by the elements $V_1^{l_1}\cdot\ldots\cdot V_m^{l_m}x$ with Maslov grading d and Alexander grading

$$A(V_1^{l_1}\cdot\ldots\cdot V_m^{l_m}x)=A(x)-\sum_{i=1}^m l_i\leqslant s\,.$$

8.1.2 The differential

First we take $x,y\in S(D)$. The set $\mathrm{Rect}(x,y)$ is defined in the following way: it is always empty except when x and y differs only by a pair of points, say $\{a,b\}$ in x and $\{c,d\}$ in y; then $\mathrm{Rect}(x,y)$ consists of the two rectangles in the torus represented by D that have bottom-left and top-right vertices in $\{a,b\}$ and bottom-right and top-left vertices in $\{c,d\}$. We call $\mathrm{Rect}^\circ(x,y)\subset\mathrm{Rect}(x,y)$ the subset of the rectangles which do not contain a point of x (or y) in their interior, that are called the *empty rectangles*.

The differential $\widehat{\partial}$ is defined as follows:

$$\widehat{\partial}x=\sum_{y\in S(D)}\sum_{\substack{r\in\mathrm{Rect}^\circ(x,y)\\ r\cap s\mathbb{O}=\varnothing}}V_1^{O_1(r)}\cdot\ldots\cdot V_m^{O_m(r)}y\quad\text{for any } x\in S(D)$$

where $O_i(r)=\begin{cases}1 & \text{if } O_i\in r\\ 0 & \text{if } O_i\notin r\end{cases}$. Here $\{O_1,\ldots,O_m\}$ is the set of the $m=\mathrm{grd}(D)-n$ normal O-markings.

We extend $\widehat{\partial}$ to $\widehat{GC}_d(D)$ linearly, and we call it $\widehat{\partial}_d$, then again to the whole $\widehat{GC}(D)$ in the following way: $\widehat{\partial}(V_ix)=V_i\cdot\widehat{\partial}x$ for every $i=1,\ldots,m$ and $x\in S(D)$. Since $\widehat{\partial}$ keeps the filtration and drops the Maslov grading by 1, see [67], we have maps

$$\widehat{\partial}_{d,s}:\mathcal{F}^s\widehat{GC}_d(D)\longrightarrow\mathcal{F}^s\widehat{GC}_{d-1}(D)$$

where $\widehat{\partial}_{d,s}$ is the restriction of $\widehat{\partial}_d$ to the subspace $\mathcal{F}^s\widehat{GC}_d(D)\subset\widehat{GC}_d(D)$. Furthermore, we have $\widehat{\partial}\circ\widehat{\partial}=0$, see [67] fo more details.

8.1.3 The homology

We define the homology group $\widehat{\mathcal{GH}}_d(D)$ as the quotient space $\dfrac{\mathrm{Ker}\,\widehat{\partial}_d}{\mathrm{Im}\,\widehat{\partial}_{d+1}}$. Moreover, we introduce the subspaces $\mathcal{F}^s\widehat{\mathcal{GH}}_d(D)$ as in Subsection 1.4.1: consider the projection $\pi_d:\mathrm{Ker}\,\widehat{\partial}_d\to\widehat{\mathcal{GH}}_d(D)$. Since $\mathrm{Ker}\,\widehat{\partial}_{d,s}=\mathrm{Ker}\,\widehat{\partial}_d\cap\mathcal{F}^s\widehat{GC}_d(D)$ we say that

$$\mathcal{F}^s\widehat{\mathcal{GH}}_d(D)=\pi_d(\mathrm{Ker}\,\widehat{\partial}_{d,s})$$

for every $s\in\mathbb{Z}$. $\mathrm{Ker}\,\widehat{\partial}_{d,s}\subset\mathrm{Ker}\,\widehat{\partial}_{d,s+1}$ implies that the filtration $\mathcal{F}$ descends to homology. We see immediately that each $\mathcal{F}^s\widehat{\mathcal{GH}}_d(D)$ is a finite dimensional $\mathbb{F}$-vector space.

We can extend the filtration $\mathcal{F}$ on the total homology

$$\widehat{\mathcal{GH}}(D)=\bigoplus_{d\in\mathbb{Z}}\widehat{\mathcal{GH}}_d(D)$$

by taking

$$\mathcal{F}^s\widehat{\mathcal{GH}}(D)=\bigoplus_{d\in\mathbb{Z}}\mathcal{F}^s\widehat{\mathcal{GH}}_d(D)\,.$$

From [67] we know that the dimension of $\mathcal{F}^s\widehat{\mathcal{GH}}_d(D)$ as an $\mathbb{F}$-vector space is a link invariant for every $d, s \in \mathbb{Z}$, in particular it is independent of the choice of the special O-markings and the ordering of the markings. Hence we can denote them with $\mathcal{F}^s\widehat{\mathcal{GH}}_d(L)$. Furthermore, [67] also tells us that $[V_i p] = [0]$ for every $i = 1, ..., m$ and $[p] \in \widehat{\mathcal{GH}}(L)$. This means that each homology class can be represented by a combination of grid states and every level $\mathcal{F}^s\widehat{\mathcal{GH}}(L)$ is also a finite dimensional $\mathbb{F}$-vector space.

The homology group $\widehat{GH}_{d,s}(L)$ of [67] can be recovered from the complex $\widehat{GC}(D)$ in the following way. We denote the graded object associated to a filtered complex $\mathcal{C}$, defined in Subsection 1.4.1, as $\left(\mathrm{gr}(\mathcal{C}), \mathrm{gr}(\partial)\right)$. Then we have that

$$\widehat{GH}_{d,s}(L) \cong_{\mathbb{F}} H_{d,s}\left(\mathrm{gr}\left(\widehat{GC}(D)\right), \mathrm{gr}(\widehat{\partial})\right) .$$

8.1.4 Equivalence between $\widehat{GH}$ and $\widehat{HFL}$

We want to show that the chain complex $\left(\widehat{GC}(D), \widehat{\partial}_1\right)$, obtained from a grid diagram D which represents the link L, can be identified with the chain complex $\left(\widehat{CFL}(H), \widehat{\partial}_2\right)$ given by an appropriate Heegaard diagram H.

Let H_D be the (unbalanced) multi-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$, see Subsection 4.1.2, defined as follows. The surface Σ is the torus which is given by D; in fact, in Subsection 8.1.1 we defined D precisely as a square representing the fundamental domain of a torus. The curves α and β correspond one-to-one with the horizontal and vertical curves in D. Finally, we take the O-markings as the set of basepoints w and the X-markings as z. Then we have the following theorem.

Theorem 8.1.1 *There is an isomorphism of chain complexes*

$$\Xi : \widehat{GC}(D) \longrightarrow \widehat{CFL}(H_D)$$

which preserves the bigradings.

Proof. We give an idea of the proof, which can be found in [67]. First, we observe that grid states in D correspond to the intersction points in H_D. Now suppose that $\phi \in \pi_2(x, y)$ is such that its associated domain in D belongs to $\mathrm{Rect}^\circ(x, y)$. Then we have that $\mu(\phi) = 1$ and $\left|\widehat{\mathcal{M}}(\phi)\right| = 1$. This is verified using the following interpretation of holomorphic disks in the symmetric product. A holomorphic disk in $\mathrm{Sym}^d(\Sigma)$ can be viewed as the following collection (F, P, f) of data:

1. a surface with boundary F, equipped with a complex structure;
2. a degree d holomorphic map $P : F \to \mathcal{D}$, where $\mathcal{D}$ is the standard disk in $\mathbb{C}$;
3. a holomorphic map $f : F \to \Sigma$.

In this correspondence, the domain of the holomorphic disk can be thought of as the two-chain induced by the map f from F to Σ. Then the claim follows from the Riemann mapping theorem. See [67] for more details.

We say that a domain is called positive if its local multiplicities are all non-negative, and at least one of them is positive. Let $\psi \in \pi_2(x,y)$ be a positive domain. Then, there is a sequence of grid states $\{x_1,...,x_k\}$, with $x_1 = x$ and $x_k = y$, and empty rectangles $\{r_i,...,r_{k-1}\}$, with $r_i \in \text{Rect}^\circ(x_1, x_{i+1})$ for $i = 1,...,k-1$, so that $\psi = r_1 * ... * r_{k-1}$. The proof of this fact is also given in [67].

At this point, we can identify the boundary maps as follows. As we said in Chapter 5, if a domain ϕ is associated to a holomorphic strip then it has non-negative local multiplicities everywhere. Domains with zero local multiplicities everywhere correspond to constant maps, which have $\mu(\phi) = 0$, and so they are not counted in the differential. Any other such domain ϕ can be factored as $\phi = \phi_1 * ... * \phi_m$, where each ϕ_i is an empty rectangle.

Since μ is additive under composition, and $\mu(\phi_i) = 1$ for what we said before, we conclude that $\mu(\phi) = m$. Since the holomorphic strips counted in the differential have Maslov index one, we conclude that only empty rectangles are counted in the differential. Moreover, each such rectangle really appears in the differential from the first obervation, completing the identification of $\widehat{\partial}_2$ with $\widehat{\partial}_1$.

Since the Maslov and Alexander gradings transform the same in both theories, when ϕ corresponds to an empty rectangle, it follows that the the identification of complexes respects relative bigradings. In fact, the identification respects absolute bigradings, since in both theories, the Alexander grading is normalized to be symmetric, and the Maslov grading is normalized using the total homology of the complexes. $\square$

In particular, this means that the simply blocked grid homology group $\widehat{GH}(L)$ is isomorphic to $\widehat{HFL}(L)$ as bigraded $\mathbb{F}$-vector spaces. In light of this, from now on in this chapter we always write $\widehat{HFL}(L)$ for the homology of the chain complex $\left(\widehat{GC}(D), \widehat{\partial}_1\right)$.

Furthermore, we denote with $\widehat{\mathcal{HFL}}(L)$ the filtered homology group $\widehat{\mathcal{GH}}(L)$ introduced in Subsection 8.1.3.

8.2 The invariant tau in the filtered theory

8.2.1 Definition of the invariant

Since $\mathcal{F}^{s-1}\widehat{\mathcal{HFL}}_d(L) \subset \mathcal{F}^s\widehat{\mathcal{HFL}}_d(L)$, and they are finite dimensional vector spaces, we define the function

$$T_L(d,s) = \dim_{\mathbb{F}} \frac{\mathcal{F}^s\widehat{\mathcal{HFL}}_d(L)}{\mathcal{F}^{s-1}\widehat{\mathcal{HFL}}_d(L)}$$

which clearly is still a link invariant.

Our first goal is to see what happens to this function T when we stabilize the link L, in other words when we add a disjoint unknot to L. Denote the unknot with the symbol $\bigcirc$. We claim that

$$T_{L \sqcup \bigcirc}(d,s) = T_L(d,s) + T_L(d+1,s) \quad \text{for any } d,s \in \mathbb{Z}\,. \tag{8.2}$$

We refer to Subsection 1.4.2 before for some remarks on filtered chain maps. Moreover, we define the shifted complex $\mathcal{C}\,[\![\, a,b \,]\!] = \mathcal{C}'$ as $\mathcal{F}^s\mathcal{C}'_d = \mathcal{F}^{s-b}\mathcal{C}_{d-a}$. Now, in order to prove Equation (8.2), we need the following proposition.

Proposition 8.2.1 *For any link L we have $\widehat{\mathcal{HFL}}(L \sqcup \bigcirc) \cong \widehat{\mathcal{HFL}}(L) \otimes V$, where V is the two dimensional $\mathbb{F}$-vector space with generators in grading and minimal level $(d,s) = (-1,0)$ and $(d,s) = (0,0)$.*

Proof. Take a grid diagram D for L. Then the extended diagram $\overline{D}$, obtained from D by adding one column on the left and one row on the top with a doubly-marked square in the top left, represents the link $L \sqcup \bigcirc$. The circle in the doubly-marked square is forced to be a special O-marking and we can also suppose that there is another special O-marking just below and right of it, as shown in Figure 8.2.

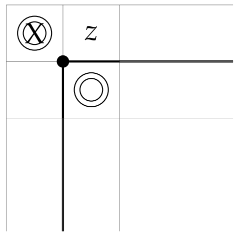


Figure 8.2: We call the top-left special O-marking O_0.

Let $I(\overline{D})$ denote the set of those generators in $S(\overline{D})$ which have a component at the lower-right corner of the doubly-marked square, and let $N(\overline{D})$ be the complement of $I(\overline{D})$ in $S(\overline{D})$. From the placement of the special O-markings, we see that $N(\overline{D})$ spans a subcomplex $\mathbf{N}$ in $\widehat{GC}(\overline{D})$. Moreover, if $\widehat{\partial}_1$ is the differential in $\widehat{GC}(\overline{D})$ and $\widehat{\partial}_2$ the one in $\widehat{GC}(D)$ we can express the restriction of $\widehat{\partial}_1$ to the subspace $\mathbf{I}$, spanned by $I(\overline{D})$, with $\widehat{\partial}_2 + \widehat{\partial}_{\mathbf{N}}$; this because there is a one-to-one correspondence between elements of $I(\overline{D})$ and grid states in $S(D)$. This correspondence induces a filtered quasi-isomorphism $i : (\mathbf{I}, \widehat{\partial}_2) \to \widehat{GC}(D)$.

Define a map $H : \mathbf{N} \to \mathbf{I}$ by the formula

$$H(x) = \sum_{y \in I(\overline{D})} \sum_{\substack{r \in \mathrm{Rect}^\circ(x,y) \\ O_0 \in r}} V_1^{O_1(r)} \cdot \ldots \cdot V_m^{O_m(r)} y \quad \text{for any } x \in N(\overline{D})\,.$$

We have that H is a filtered chain homotopy equivalence between $(\mathbf{N}, \widehat{\partial}_1)$ and $(\mathbf{I}, \widehat{\partial}_2)$, which increases the Maslov grading by one, and $H \circ \widehat{\partial}_{\mathbf{N}} = 0$. To see the first claim, we mark the square just on the right of O_0 with z and we define an operator $\widehat{H}_z : \mathbf{I} \to \mathbf{N}$, which counts only rectangles that contain z. This operator is a homology inverse of H; this and the second claim can be proved in the same way as in [67]. Those two facts together tell us that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{I} \arrow[r, "\widehat{\partial}_{\mathbf{N}}"] \arrow[d, "i"'] & \mathbf{N} \arrow[d, "i \circ H"] \\
\widehat{GC}(D) \arrow[r, "0"] & \widehat{GC}(D)\llbracket -1,0 \rrbracket
\end{tikzcd}$$

Since H is a filtered chain homotopy equivalence, $i \circ H$ is a filtered quasi-isomorphism, just like the map i. Therefore, we can use Lemma 1.4.4 and obtain that the map between the mapping cones

$$\text{Cone}(\widehat{\partial}_{\mathbf{N}}) = \widehat{GC}(\overline{D}) \longrightarrow \text{Cone}(0) = \widehat{GC}(D) \otimes V$$

is a filtered quasi-isomorphism and so the claim follows easily from Propositions 1.4.1 and 1.4.3. See also [67] for more details. □

Equation (8.2) is obtained immediately from Proposition 8.2.1, in fact we have proved that $\mathcal{F}^s\widehat{\mathcal{HFL}}_d(L \sqcup \bigcirc) \cong \mathcal{F}^s\widehat{\mathcal{HFL}}_d(L) \oplus \mathcal{F}^s\widehat{\mathcal{HFL}}_{d+1}(L)$ for every $d, s \in \mathbb{Z}$ and so it is enough to apply the definition of T.

Now we are able to do some computations. The homology of the unknot can be easily computed by taking the grid diagram of dimension one, where the square is marked with both X and O. The complex has one element of Maslov and Alexander grading 0; then $T_{\bigcirc}(d, s) = 1$ if $(d, s) = (0, 0)$ and it is 0 otherwise.

Using Equation (8.2) we get the function T of the n-component unlink $\bigcirc_n$:

$$T_{\bigcirc_n}(d, s) = \begin{cases} \binom{n-1}{k} & \text{if } (d, s) = (-k, 0), \quad 0 \leqslant k \leqslant n-1 \\ 0 & \text{otherwise} \end{cases} .$$

We can also see this directly from Proposition 8.2.1, in fact we have that $\widehat{\mathcal{HFL}}(\bigcirc_n) \cong V^{\otimes(n-1)}$.

Now let us consider a grid diagram D of a link L. The Maslov grading of the elements of $S(D)$ and the differential $\widehat{\partial}$ are independent of the position of the X's, once we have fixed the special O-markings. Since we can always change the X-markings to obtain $\bigcirc_n$, this means that $\dim_{\mathbb{F}}\widehat{\mathcal{HFL}}_d(L) = \dim_{\mathbb{F}}\widehat{\mathcal{HFL}}_d(\bigcirc_n)$ for every $d \in \mathbb{Z}$ and the generators are the same. In particular

$$\widehat{\mathcal{HFL}}(L) \cong_{\mathbb{F}} \widehat{\mathcal{HFL}}(\bigcirc_n) \cong_{\mathbb{F}} \mathbb{F}^{2^{n-1}} \quad \text{and} \quad \widehat{\mathcal{HFL}}_d(L) \cong_{\mathbb{F}} \mathbb{F}^{\binom{n-1}{-d}} \quad \text{when } 1-n \leqslant d \leqslant 0\,.$$

From this we have that $\widehat{\mathcal{HFL}}_0(L)$ has always dimension one and then we define $\tau(L)$ as the only integer s such that $T_L(0, s) > 0$. We remark that for a knot this version of τ coincides with the one of Ozsváth and Szabó. See the proof of Theorem 8.4.1 in Section 8.4. We also observe that Equation (8.2) tells that $\tau(L \sqcup \bigcirc) = \tau(L)$.

8.2.2 Dropping the special O-markings

In this subsection we study what happens to the homology of the filtered chain complex $\left(\widehat{GC}(D), \widehat{\partial}\right)$ if the grid diagram D has less than n special O-markings.

Let us consider D a grid diagram for an n-component link L. The set $s\mathbb{O} \subset \mathbb{O}$ contains at most one O-marking from each component of L, but we have that $|s\mathbb{O}| \geqslant 1$. Denote with $m = \text{grd}(D) - |s\mathbb{O}|$ the number of normal O-markings in D. Then we can define our chain complex exactly in the same way as in Section 8.1; on the other hand, the homology $\widehat{\mathcal{GH}}(D)$ is no longer a link invariant, in fact it clearly depends of the choice of the special O-markings.

Nonetheless, we can show that the $\mathbb{F}$-vector space $\widehat{\mathcal{GH}}(D)$ is still finite dimensional. Note that this is not true if instead we consider the bigraded simply blocked grid homology group $\widehat{GH}(D)$.

Proposition 8.2.2 *The homology group $\widehat{\mathcal{GH}}(D)$, defined as in Subsection 8.1.3, is $\mathbb{F}$-isomorphic to $\widehat{\mathcal{GH}}\left(\bigcirc_{|s\mathbb{O}|}\right)$, the homology of the unlink with $|s\mathbb{O}|$ components each containing a special O-marking. In particular, we have that*

$$\widehat{\mathcal{GH}}(D) \cong_{\mathbb{F}} \mathbb{F}^{2^{|s\mathbb{O}|-1}} \quad \textit{and} \quad \widehat{\mathcal{GH}}_d(D) \cong_{\mathbb{F}} \mathbb{F}^{\binom{|s\mathbb{O}|-1}{-d}}$$

when $1 - |s\mathbb{O}| \leqslant d \leqslant 0$.

Proof. As we noted before, the group $\widehat{\mathcal{GH}}(D)$ does not depend on the position of the X-marking. Since we can always change them in a way that D becomes a diagram for an unlink with a special O-marking on every component, the claim follows from the results in the previous subsection. □

Even though in this case the homology is no longer a link invariant, we can still prove the following theorem.

Theorem 8.2.3 *Let us consider two grid diagrams D_1 and D_2 representing smoothly isotopic links L_1 and L_2 such that all the isotopic components both contain or not contain a special O-marking. Then we have that $\widehat{\mathcal{GH}}(D_1)$ is filtered isomorphic to $\widehat{\mathcal{GH}}(D_2)$ and the isomorphism preserves the Maslov grading.*

Proof. From [82] we know that such two grid diagrams differ by a finite sequence of grid moves: reordering of the O-markings, commutations and stabilizations. Then it is enough to prove the theorem in the case when D_2 is obtained from D_1 by one of these three moves.

The maps that we defined in Chapter 5 can be arranged to give filtered quasi-isomorphisms for each move. This implies that $\widehat{\mathcal{GH}}(D_1) \cong \widehat{\mathcal{GH}}(D_2)$. See also the results in [67] to find the maps in the grids setting. □

Hence, we can denote the homology group of an n-component link L with $\widehat{\mathcal{GH}}^{\mathbb{O}}(L)$ and it depends only on which components of the link contain a special O-marking. We use the homology groups $\widehat{\mathcal{GH}}^{\mathbb{O}}(L)$ to define some cobordism maps in Section 8.3.

8.2.3 Symmetries

Reversing the orientation

If $-L$ is the link obtained from L by reversing the orientation of all the components then

$$T_{-L}(d,s) = T_L(d,s) \quad \text{for any } d,s \in \mathbb{Z} \tag{8.3}$$

and $\tau(-L) = \tau(L)$.

To see this, consider a grid diagram D of L, then it is easy to observe that, if we reflect D along the diagonal going from the top-left to the bottom-right of the grid, the diagram D' obtained in this way represents $-L$. Hence, we take the map $\Phi : S(D) \to S(D')$ that sends a grid state x into its reflection x^- and now, from [67], we have that $M(x^-) = M(x)$ and $A(x^-) = A(x)$. This means that Φ is a filtered quasi-isomorphism between $\widehat{GC}(D)$ and $\widehat{GC}(D')$, since clearly the differentials commute with Φ.

This gives another proof that $\widehat{\mathcal{HFL}}(-L) \cong \widehat{\mathcal{HFL}}(L)$ and then Equation (8.3) follows.

Mirror image

For an n-component link L we have that the function T of the mirror image L^* is given by the following equation

$$T_{L^*}(d,s) = T_L(-d+1-n,-s) \quad \text{for any } d,s \in \mathbb{Z}\,. \tag{8.4}$$

To see this, we first, given a complex $\mathcal{C}$ with a filtration $\mathcal{F}$, introduce the filtered dual complex $\mathcal{C}^*$, equipped with a filtration $\mathcal{F}^*$ by taking

$$(\mathcal{F}^*)^s(\mathcal{C}^*)_d = \mathrm{Ann}(\mathcal{F}^{-s-1}\mathcal{C}_{-d}) \subset (\mathcal{C}_{-d})^* = (\mathcal{C}^*)_d \quad \text{for any } d,s \in \mathbb{Z}\,,$$

where $\mathrm{Ann}(\mathcal{F}^h\mathcal{C}_k)$ is the subspace of $(\mathcal{C}_k)^*$ consisting of all the linear functionals that are zero over $\mathcal{F}^h\mathcal{C}_k$.

Second, given a grid diagram D of L, we call $\left(\widetilde{GC}(D),\widetilde{\partial}\right)$ the filtered chain complex $\dfrac{\left(\widehat{GC}(D),\widehat{\partial}\right)}{V_1=\ldots=V_m=0}$ and we also denote with W the two dimensional $\mathbb{F}$-vector space with generators in grading and minimal level $(d,s)=(0,0)$ and $(d,s)=(-1,-1)$.

Lemma 8.2.4 *We have the filtered quasi-isomorphism*

$$\widetilde{GC}(D) \cong \widehat{GC}(D) \otimes W^{\otimes(grd(D)-n)}\,, \tag{8.5}$$

where D is a grid diagram for an n-component link L.

Proof. Suppose that V_1 and V_2 are variables which belong to the same component of L. Then it follows from [67, Lemma 14.1.11] that

$$\frac{\widehat{GC}(D)}{V_1=V_2} \cong \widehat{GC}(D) \otimes W\,.$$

Therefore, we conclude by iterating this procedure. □

We want to prove the following proposition.

Proposition 8.2.5 $\widehat{\mathcal{HFL}}(L^*) \cong \widehat{\mathcal{HFL}}^*(L)[\![1-n,0]\!]$*, where the filtration on* $\widehat{\mathcal{HFL}}^*(L)$ *is* $\mathcal{F}^*$*.*

Proof. Let D^* be the diagram obtained by reflecting D through a horizontal axis. The diagram D^* represents L^*. Reflection induces a bijection $x \to x^*$ between grid states for D and those for D^*, inducing a bijection between empty rectangles in $\mathrm{Rect}^\circ(x,y)$ and empty rectangles in $\mathrm{Rect}^\circ(y^*,x^*)$. Hence, the reflection induces a filtered isomorphism

$$\widetilde{GC}(D^*) \cong \widetilde{GC}^*(D)\,[\![\,1-\mathrm{grd}(D), n-\mathrm{grd}(D)\,]\!]\,,$$

where the shifts are given by the fact that $M(x^*) = -M(x)+1-\mathrm{grd}(D)$ and $A(x^*) = -A(x)+n-\mathrm{grd}(D)$.

Now Lemma 8.2.4 and observing that

$$(W^*)^{\otimes(\mathrm{grd}(D)-n)} \cong W^{\otimes(\mathrm{grd}(D)-n)}\,[\![\,\mathrm{grd}(D)-n, \mathrm{grd}(D)-n\,]\!]$$

lead to the filtered quasi-isomorphism

$$\widehat{GC}(D^*) \cong \widehat{GC}^*(D)\,[\![\,1-n,0\,]\!]\,.$$

□

Proposition 8.2.5 says that $\mathcal{F}^s\widehat{\mathcal{HFL}}_d(L^*) \cong (\mathcal{F}^*)^s \left(\widehat{\mathcal{HFL}}^*\right)_{d-1+n}(L)$ for every $d,s \in \mathbb{Z}$. Then we can prove Equation (8.4):

$$T_{L^*}(d,s) = \dim \frac{(\mathcal{F}^*)^s \left(\widehat{\mathcal{HFL}}^*\right)_{d-1+n}(L)}{(\mathcal{F}^*)^{s-1} \left(\widehat{\mathcal{HFL}}^*\right)_{d-1+n}(L)} = \dim \frac{\operatorname{Ann}\left(\mathcal{F}^{-s-1}\widehat{\mathcal{HFL}}_{-d+1-n}(L)\right)}{\operatorname{Ann}\left(\mathcal{F}^{-s}\widehat{\mathcal{HFL}}_{-d+1-n}(L)\right)} =$$
$$= \dim \frac{\mathcal{F}^{-s}\widehat{\mathcal{HFL}}_{-d+1-n}(L)}{\mathcal{F}^{-s-1}\widehat{\mathcal{HFL}}_{-d+1-n}(L)} = T_L(-d+1-n,-s)\,.$$

If we define $\tau^*(L)$ as the unique integer such that $T_L(1-n,\tau^*(L)) = 1$ then we have proved that

$$\tau(L^*) = -\tau^*(L)\,. \tag{8.6}$$

In particular for a knot K, where $\tau^*(K) = \tau(K)$, we have $\tau(K^*) = -\tau(K)$. Moreover, we have the following corollary.

Corollary 8.2.6 *Suppose that L is smoothly isotopic to L^*. Then,*

$$T_{L^*}(d,s) = T_L(d,s) \quad \textit{for any } d,s \in \mathbb{Z}$$

and so Equation (8.4) *gives that the function T_L has a central symmetry in the point $\left(\frac{1-n}{2},0\right)$. In particular $\tau^*(L) = -\tau(L)$ and, for knots, $\tau(K) = 0$.*

Connected sum

Given two links L_1 and L_2, the function T of the connected sum $L_1\# L_2$ is the convolution product of the T functions of L_1 and L_2; in other words

$$T_{L_1\# L_2}(d,s) = \sum_{\substack{d=d_1+d_2\\ s=s_1+s_2}} T_{L_1}(d_1,s_1)\cdot T_{L_2}(d_2,s_2) \quad \text{for any } d,s \in \mathbb{Z}\,. \tag{8.7}$$

This equation is very hard to prove in the grid diagram settings, but it can be proved using the holomorphic definition of link Floer Homology, as we saw in Subsection 6.6.1. See also [72].

We see immediately that the homology and the T function of $L_1\# L_2$ are independent of the choice of the components used to perform the connected sum; moreover, the τ-invariant is additive:

$$\tau(L_1\# L_2) = \tau(L_1) + \tau(L_2)\,.$$

Disjoint union

The disjoint union of two links L_1 and L_2 is equivalent to $L_1\#\,(L_2 \sqcup \bigcirc)$. Thus by Equations (8.2) and (8.7) we have the following relation:

$$T_{L_1\sqcup L_2}(d,s) = \sum_{\substack{d=d_1+d_2\\ s=s_1+s_2}} T_{L_1}(d_1,s_1)\cdot \left(T_{L_2}(d_2,s_2) + T_{L_2}(d_2+1,s_2)\right) \quad \text{for any } d,s \in \mathbb{Z}\,, \tag{8.8}$$

or in other words $\widehat{\mathcal{HFL}}(L_1 \sqcup L_2) \cong \widehat{\mathcal{HFL}}(L_1) \otimes \widehat{\mathcal{HFL}}(L_2) \otimes V$, where V is the two dimensional $\mathbb{F}$-vector space with generators in grading and minimal level $(d,s) = (-1,0)$ and $(d,s) = (0,0)$. We have immediately that

$$\tau(L_1 \sqcup L_2) = \tau(L_1\# L_2) = \tau(L_1) + \tau(L_2)\,.$$

Quasi-alternating links

We recall that quasi-alternating links are the smallest set of links $\mathcal{Q}$ that satisfies the two properties:

1. the unknot is in $\mathcal{Q}$;
2. L is in $\mathcal{Q}$ if it admits a diagram with a crossing whose two resolutions L_0 and L_1 are both in $\mathcal{Q}$, $\det(L_i) \neq 0$ and $\det(L_0) + \det(L_1) = \det(L)$.

The above definition and a result in [71] imply that every quasi-alternating link is non-split and every non-split alternating link is quasi-alternating. Moreover, quasi-alternating links are both Khovanov and link Floer homology thin, which means that their homologies are supported in two and one lines respectively, and the homology is completely determined by the signature and the Jones (Alexander in the hat version of link Floer homology) polynomial. The following proposition says that the same is true in filtered grid homology.

Theorem 8.2.7 *If L is an n-component quasi-alternating link then the function T_L is supported in a line; more specifically the following relation holds:*

$$T_L(d,s) \neq 0 \quad \textit{if and only if} \quad s = d + \frac{n-1-\sigma(L)}{2} \quad \textit{for } 1-n \leqslant d \leqslant 0$$

where $\sigma(L)$ is the signature of L.

Proof. We already know that if $T_L(d,s) \neq 0$ then $1-n \leqslant d \leqslant 0$, so we only have to prove the alignment part of the statement.

Take a grid diagram D for L, then from [67] the claim is true for the bigraded homology of $\left(\mathrm{gr}\left(\widehat{GC}(D)\right), \mathrm{gr}(\widehat{\partial})\right) \cong \widehat{HFL}(L)$. Since $T_L(d,s) \neq 0$ implies that $H_{d,s}\left(\mathrm{gr}\left(\widehat{GC}(D)\right), \mathrm{gr}(\widehat{\partial})\right)$ is non-zero, the theorem follows. □

From Theorem 8.2.7 we obtain immediately the following corollary.

Corollary 8.2.8 *If L is an n-component quasi-alternating link then $\tau(L) = \frac{n-1-\sigma(L)}{2}$ and $\tau(L^*) = n-1-\tau(L)$.*

8.3 Cobordisms

8.3.1 Induced maps and degree shift

In this section we study the behaviour of the function T under cobordisms, see Section 1.3. Some of the induced maps that appear in this subsection come from the work of Sarkar in [82]; though the grading shifts are different, because Sarkar used a different definition of the Alexander grading, ignoring the number of component of the link.

It is a standard result in Morse theory that a link cobordism can be decomposed into five standard cobordisms. We find maps in homology for each case. From now on, given a link L_i, we denote with D_i one of its grid diagrams.

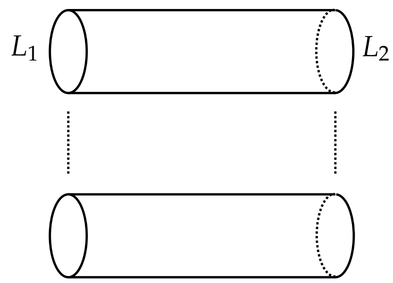


Figure 8.3: Identity cobordism.

i) *Identity cobordism*. This cobordism, with no critical points (Figure 8.3), represents a sequence of Reidemeister moves; in other words L_1 and L_2 are smoothly isotopic. At the end of Section 8.1 we remarked that filtered homology is a link invariant; more precisely what we have is a filtered quasi-isomorphism between $\widehat{GC}(D_1)$ and $\widehat{GC}(D_2)$. This, as we know, induces a filtered isomorphism in homology.

ii) *Split cobordism*. This cobordism (right in Figure 8.4) represents a band move when L_2 has one more component than L_1. Take D_1 with a 2×2 square with two X-markings, one at the top-left and one at the bottom-right; then we claim that D_2 is obtained from D_1 by deleting this two X-markings and putting two new ones: at the top-right and the bottom-left, as shown in Figure 8.5. In order to construct the complex $\widehat{GC}(D_2)$ we

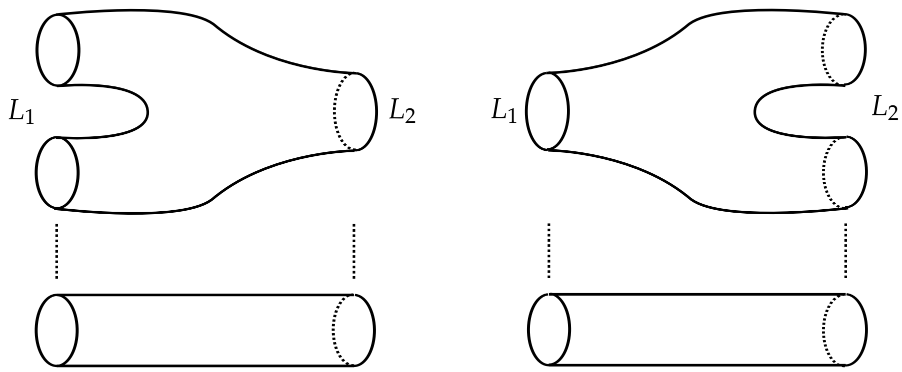


Figure 8.4: Merge and split cobordisms.

need to create one more special O-marking on the new component of L_2. To avoid this problem we first consider the identity map in the filtered $\widetilde{GC}$ theory

$$\mathrm{Id} : \widetilde{GC}(D_1) \longrightarrow \widetilde{GC}(D_2)\,,$$

which clearly is a chain map since now every O-marking is special; moreover, it induces an isomorphism in homology, that preserves the Maslov grading, and a direct computation gives that it is filtered of degree one.

Now we use Equation (8.5) and we get an isomorphism

$$\Phi_{\mathrm{Split}} : \widehat{\mathcal{HFL}}(L_1) \otimes W \longrightarrow \widehat{\mathcal{HFL}}(L_2)$$

that is a degree one filtered map which still preserves the Maslov grading. The W

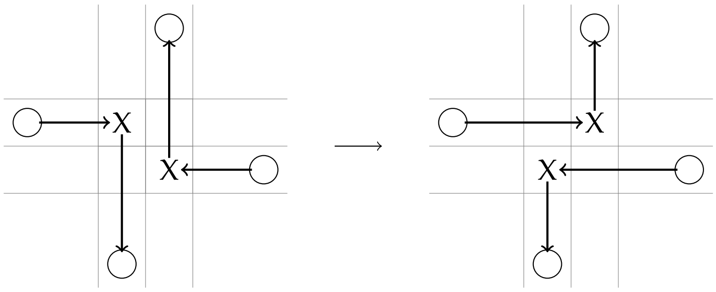


Figure 8.5: Band move in a grid diagram.

factor appears because in Equation (8.5) we take into account the size of D_i and the number of components of L_i; while the first quantity is the same for both diagrams the link L_2 has one more component than L_1.

iii) *Merge cobordism.* This cobordism (left in Figure 8.4) represents a band move when L_1 has one more component than L_2. We have an isomorphism

$$\Phi_{\mathrm{Merge}} : \widehat{\mathcal{HFL}}(L_1) \longrightarrow \widehat{\mathcal{HFL}}(L_2) \otimes W\,.$$

The map is obtained in the same way as Φ_{Split} in the previous case, but with the difference that now it is filtered of degree zero.

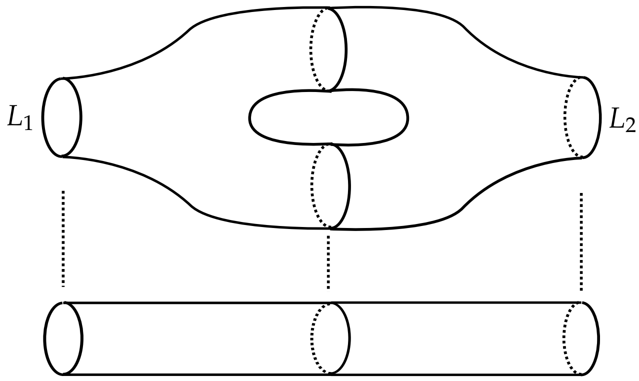


Figure 8.6: Torus cobordism.

Sometimes we are more interested in when a split and a merge cobordism appear together, the second just after the first, in the shape of what we call a torus cobordism (Figure 8.6). We have the following proposition.

Proposition 8.3.1 *Let Σ be a torus cobordism between two links L_1 and L_2. Then Σ induces a Maslov grading preserving isomorphism between $\widehat{\mathcal{HFL}}(L_1)$ and $\widehat{\mathcal{HFL}}(L_2)$, which is filtered of degree one.*

Proof. We can choose D_1 in a way that the split and the merge band moves can be performed on two disjoint bands. Then we apply twice the move shown in Figure 8.5 and we take as map the identity. In this case the identity is a chain map because L_2 has the same number of components of L_1; this means that the special O-markings in D_1 and D_2 are the

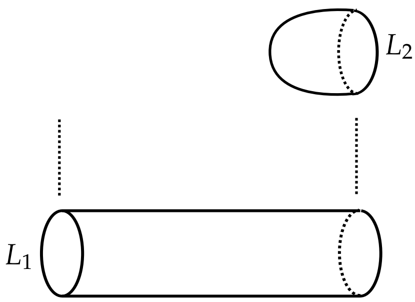


Figure 8.7: Birth cobordism.

same and then the two differentials coincide. In this way we obtain an isomorphism in homology with Maslov grading shift and filtered degree equal to the sum of the ones in ii) and iii). □

iv) *Birth cobordism.* A cobordism (Figure 8.7) representing a birth move.

Since our cobordisms have boundary in both L_1 and L_2, we can always assume that a birth move is followed (possibly after some Reidemeister moves) by a merge move. Thus it is enough to define a map for the composition of these three cobordisms and this is what we do in the following proposition.

Proposition 8.3.2 *Let Σ be a cobordism between two links L_1 and L_2 like the one in Figure 8.8. Then Σ induces an isomorphism Φ_{Birth} between $\widehat{\mathcal{HFL}}(L_1)$ and $\widehat{\mathcal{HFL}}(L_2)$ that preserves the Maslov grading and it is filtered of degree zero.*

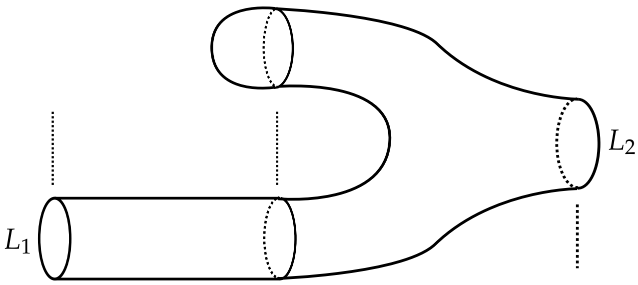


Figure 8.8: A more useful birth cobordism.

Proof. The first step is to construct a map s_1 associated to the grid move shown in Figure 8.9; note that we add a normal O-marking, since later we merge the new unknot component with an already exisiting one. Let us denote with D_1' the stabilized diagram and with $c = \alpha \cap \beta$ the point in the picture; then we have the inclusion $i : S(D_1) \to S(D_1')$ that sends a grid state x in D_1 to the grid state in D_1' constructed from x by adding the point c. Then $s_1 : \widehat{GC}(D_1) \to \widehat{GC}(D_1')$ is defined by the following formula:

$$s_1(x) = \sum_{y \in S(D_1')} \sum_{\substack{H \in \mathcal{SL}(i(x),y,c) \\ H \cap s\mathbb{O} = \emptyset}} V_1^{n_1(H)} \cdot \ldots \cdot V_m^{n_m(H)} y \quad \text{for any } x \in S(D_1)$$

where $\mathcal{SL}(x,z,p)$ is the set of all the snail-like domains, the exact definition can be found

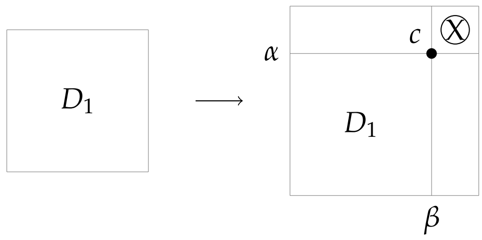


Figure 8.9: Birth move in a grid diagram.

in [67], centered at p joining x to z, illustrated in Figure 8.10; $n_i(H)$ is the number of times H passes through O_i and m is the number of the normal O-markings of D_1. In [58] is proved that s_1 is a filtered quasi-isomorphism which induces a filtered isomorphism between $\widehat{\mathcal{HFL}}(L_1) \to \widehat{\mathcal{GH}}^{\mathbb{O}}(L_1 \sqcup \bigcirc)$; where the special O-markings on $L_1 \sqcup \bigcirc$ coincide with the ones on L_1 (the new unknotted component has a normal O-marking). Moreover, in [82] Sarkar showed that the map s_1 is filtered of degree zero.

At this point we compose s_1 with the map s_2 given by the Reidemeister moves, which is a filtered quasi-isomorphism by Theorem 8.2.3, and finally with s_3, the identity associated to the band move of Figure 8.5. The map s_3 induces an isomorphism $\widehat{\mathcal{GH}}^{\mathbb{O}}(L_1 \sqcup \bigcirc) \to$

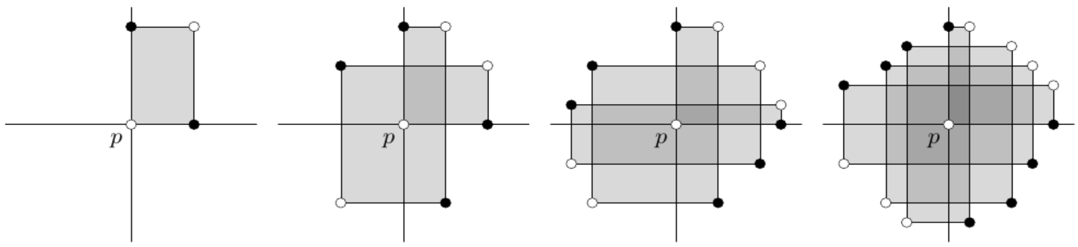


Figure 8.10: Some of the snail-like domains $\mathcal{SL}(x, z, p)$: the coordinates of x and z are represented by the white and black circles.

$\widehat{\mathcal{HFL}}(L_2)$ that clearly preserves the Maslov grading and again we easy compute that it is filtered of degree zero.

Hence, the composition of these three maps induces the isomorphism in the claim. □

v) *Death cobordism*. This cobordism (Figure 8.11) represents a death move. Since this move can also be seen as a birth move between L_2^* and L_1^*, we take the dual map of

$$\Phi_{\text{Birth}} : \widehat{\mathcal{HFL}}(L_2^*) \longrightarrow \widehat{\mathcal{HFL}}(L_1^*)$$

which exists from Proposition 8.3.2; then $\Phi^*_{\text{Birth}} = \Phi_{\text{Death}}$, by Proposition 8.2.5, is a map between $\widehat{\mathcal{HFL}}(L_1)$ and $\widehat{\mathcal{HFL}}(L_2)$. Furthermore, it is still an isomorphism that is filtered of degree zero and preserves the Maslov grading.

The results for birth and death cobordisms in this section immediately give the following corollary.

Corollary 8.3.3 *Suppose there is a birth or a death cobordism as in Figures 8.8 and 8.11 between two links L_1 and L_2. Then we have that $\widehat{\mathcal{HFL}}(L_1) \cong \widehat{\mathcal{HFL}}(L_2)$.*

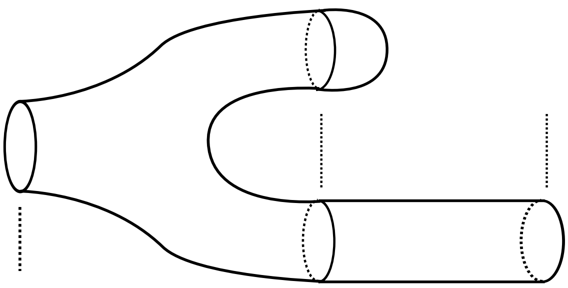


Figure 8.11: Death cobordism.

8.3.2 Strong concordance invariance

We remark that a strong cobordism is a cobordism Σ, between two links with the same number of components, such that every connected component of Σ is a knot cobordism between a component of the first link and one of the second link. Moreover, if the connected components of Σ are all annuli then we call Σ a strong concordance, see Section 1.3. We want to prove the following theorem.

Theorem 8.3.4 *The function T is a strong concordance invariant. In other words, if L_1 and L_2 are strongly concordant then $T_{L_1}(d,s) = T_{L_2}(d,s)$ for every $d,s \in \mathbb{Z}$.*

We start by observing that Proposition 8.3.1 leads to the following corollary.

Corollary 8.3.5 *Suppose there is a strong cobordism Σ between L_1 and L_2 such that Σ is the composition of $g(\Sigma)$ torus cobordisms, not necessarily all of them belonging to the same component of Σ. Then Σ induces an isomorphism between $\widehat{\mathcal{HFL}}(L_1)$ and $\widehat{\mathcal{HFL}}(L_2)$, which is filtered of degree $g(\Sigma)$ and preserves the Maslov grading.*

Now, if we have an isomorphism $F : \widehat{\mathcal{HFL}}(L_1) \to \widehat{\mathcal{HFL}}(L_2)$ that preserves the Maslov grading and it is filtered of degree t, which means that there are inclusions $F\left(\mathcal{F}^s\widehat{\mathcal{HFL}}_d(L_1)\right) \subset \mathcal{F}^{s+t}\widehat{\mathcal{HFL}}_d(L_2)$ for every $d,s \in \mathbb{Z}$, then $\tau(L_2) \leqslant \tau(L_1) + t$. Hence, we can prove the following theorem that immediately implies the invariance statement.

Theorem 8.3.6 *Suppose that Σ is a strong cobordism between two links L_1 and L_2. Then*

$$\left|\tau(L_1) - \tau(L_2)\right| \leqslant g(\Sigma)\,.$$

Furthermore, if L_1 and L_2 are strongly concordant then $\widehat{\mathcal{HFL}}(L_1) \cong \widehat{\mathcal{HFL}}(L_2)$.

Proof. From [67] we can suppose that, in Σ, 0-handles come before 1-handles while 2-handles come later; moreover, we can say that Σ is the composition of birth, torus and death cobordisms (and obviously some identity cobordisms). Each of these induces an isomorphism in homology that also respects the Maslov grading.

For the first part, we only need to check what is the filtered degree of the isomorphism between $\widehat{\mathcal{HFL}}(L_1)$ and $\widehat{\mathcal{HFL}}(L_2)$, obtained by the composition of all the induced maps on each piece of Σ. Birth, death and identity are filtered of degree zero, while, from Corollary 8.3.5, the torus cobordisms are filtered of degree $g(\Sigma)$. Then we obtain

$$\tau(L_2) \leqslant \tau(L_1) + g(\Sigma)\,.$$

For the other inequality we consider the same cobordism, but this time from L_2 to L_1.

Now, for the second part, we observe that now there are no torus cobordisms and then the claim follows from Corollary 8.3.3. □

8.3.3 A lower bound for the slice genus

Suppose Σ is a cobordism (not necessarily strong) between two links L_1 and L_2. Denote with $\Sigma_1, ..., \Sigma_J$ the connected components of Σ. For $i = 1, 2$ we define the integers $l_i^k(\Sigma)$ as the number of components of L_i that belong to Σ_k minus 1; in particular $l_i^k(\Sigma) \geqslant 0$ for any k, i. Finally, we say that $l_i(\Sigma) = \sum_{k=1}^{J} l_i^k(\Sigma) = n_i - J$, where n_i is the number of component of L_i. For example, if Σ is the cobordism in Figure 8.12 then $J = 2$, ordering Σ_1 and Σ_2 from above to bottom, we have that $l_1(\Sigma) = 3$ and $l_2(\Sigma) = 4$; while $l_1^1(\Sigma) = 2$, $l_1^2(\Sigma) = 1$, $l_2^1(\Sigma) = 3$ and $l_2^2(\Sigma) = 1$. We have the following lemma.

Lemma 8.3.7 *If Σ has no 0,2-handles then, up to rearranging 1-handles, we can suppose that Σ is like in Figure 8.12: there are $l_1(\Sigma)$ merge cobordisms between $(0, t_1)$, $l_2(\Sigma)$ split cobordisms between $(t_2, 1)$ and $g(\Sigma)$ torus cobordisms between (t_1, t_2). We have no other 1-handles except for the ones we considered before.*

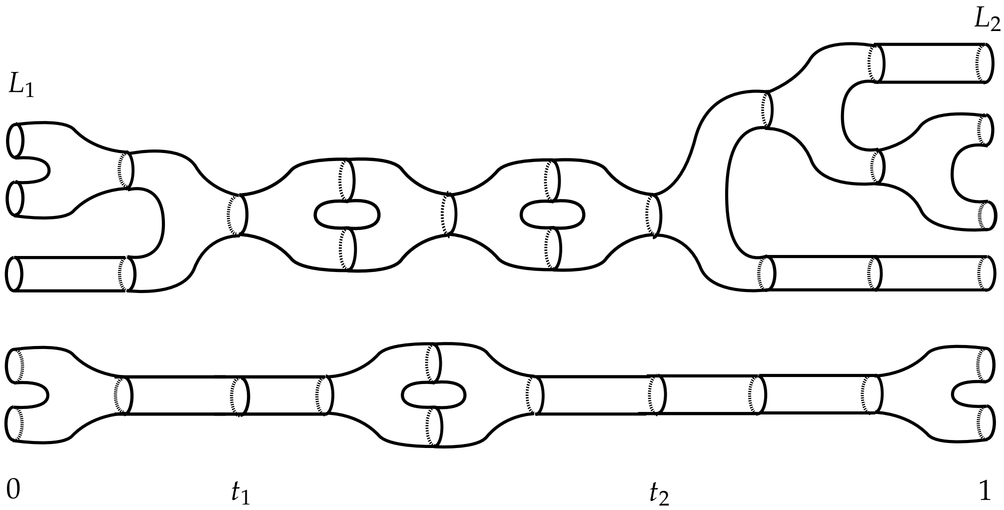


Figure 8.12: An example of a cobordism between two links after rearranging handles.

Proof. We consider a connected component Σ_k, which is a cobordism between L_1^k and L_2^k, and we fix a Morse function $f : S^3 \times [0, 1] \to [0, 1]$. After some Reidemeister moves, from [67] we know that we can assume that all the band moves are performed on disjoint bands, in particular we can apply them in every possible order.

Since Σ_k has boundary in both L_1^k and L_2^k by the definition of cobordism given at beginning of Section 1.3, if we take an ordering for the components of L_1^k, we can find a merge cobordism joining the first and the second component of L_1^k at some point $\bar{t}$ in $(0, 1)$; we assume that the associated band move is the first we apply on L_1^k. Now we just do the same thing on the other components, but taking the new component instead of the first two. In this way we have that there is a $t_1 \in (0, 1)$ such that $\Sigma_k \cap f^{-1}[0, t_1]$ is composed by $l_1^k(\Sigma)$ merge cobordisms and $\Sigma_k \cap f^{-1}(t_1)$ is a knot.

In the same way we find that, for a certain $t_2 \in (0, 1)$, the cobordism $\Sigma_k \cap f^{-1}[t_2, 1]$ is composed by $l_2^k(\Sigma)$ split cobordisms and $\Sigma_k \cap f^{-1}(t_2)$ is a knot.

At this point $\Sigma_k \cap f^{-1}[t_1, t_2]$ is a knot cobordism of genus $g(\Sigma_k)$ and from [67], see also [51], we can rearrange the saddles to obtain a composition of $g(\Sigma_k)$ torus cobordisms like in Figure 8.12.

To see that there are no other 1-handles left it is enough to compute the Euler characteristic of Σ_k:

$$2 - 2g(\Sigma_k) - l_1^k(\Sigma) - 1 - l_2^k(\Sigma) - 1 = \chi(\Sigma_k) = -|\text{1-handles}| \,.$$

This means that the number of 1-handles in Σ_k is precisely $2g(\Sigma_k) + l_1^k(\Sigma) + l_2^k(\Sigma)$. □

From Figure 8.12 we realize that merge and split cobordisms can appear alone in Σ and not always in pair like in strong cobordisms. In case ii) and iii) of Subsection 8.3.1 we see that they do not induce isomorphisms in homology, but we find maps Φ_{Split} and Φ_{Merge} that are indeed isomorphisms if restricted to $\widehat{\mathcal{HFL}}_0(L_1) \to \widehat{\mathcal{HFL}}_0(L_2)$; moreover, the filtered degree is one for split cobordisms and zero for merge cobordisms. Since we are looking for information on τ, this is enough for our goal and then we can prove the following inequality.

Proposition 8.3.8 *Suppose Σ is a cobordism between two links L_1 and L_2. Then*

$$\left|\tau(L_1) - \tau(L_2)\right| \leqslant g(\Sigma) + max\left\{l_1(\Sigma), l_2(\Sigma)\right\} .$$

Proof. From Corollary 8.3.3 and Theorem 8.3.6, we can suppose that there are no 0 and no 2-handles in Σ. We can also assume that Σ is like in Lemma 8.3.7.

All of these cobordisms induce isomorphisms of the homology in Maslov grading zero. The number of torus cobordisms is $g(\Sigma)$ while the number of split cobordisms (that are not part of torus cobordisms) is $l_2(\Sigma)$. This means that

$$\tau(L_2) \leqslant \tau(L_1) + g(\Sigma) + l_2(\Sigma) \,.$$

Now we do the same, but considering the cobordism going from L_2 to L_1, as we did in the proof of Theorem 8.3.6. We obtain that

$$\tau(L_1) \leqslant \tau(L_2) + g(\Sigma) + l_1(\Sigma) \,.$$

Putting the two inequalities together proves the relation in the statement of the theorem. □

If L is an n-component link, from Proposition 8.3.8 we have immediately that

$$|\tau(L)| + 1 - n \leqslant g_4(L) \tag{8.9}$$

which, as we already said, is a lower bound for the slice genus of our link. Indeed, we can say more by using Equation (8.6) and observing that $g_4(L^*) = g_4(L)$:

$$\max\left\{|\tau(L)|, |\tau^*(L)|\right\} + 1 - n \leqslant g_4(L) \,.$$

8.4 The $\mathcal{HFL}^-$ version of the homology

8.4.1 A different point of view

The collapsed filtered homology $c\mathcal{HFL}^-$ can also be obtained from a grid diagram D as the free $\mathbb{F}[V_1, ..., V_{\mathrm{grd}(D)-n}, V]$-module generated by the set of grid states $S(D)$. This ring has one more variable V, compared to the ring we considered for $\widehat{GC}(D)$, associated to the special O-markings. The differential ∂^- is defined as following:

$$\partial^- x = \sum_{y \in S(D)} \sum_{r \in \mathrm{Rect}^\circ(x,y)} V_1^{O_1(r)} \cdot ... \cdot V_m^{O_m(r)} \cdot V^{O(r)} y \quad \text{for any } x \in S(D)$$

where $m = \mathrm{grd}(D) - n$ and $O(r)$ is the number of special O-markings in r.

It is clear from the definition that

$$\left(\widehat{GC}(D), \widehat{\partial}\right) = \frac{\left(cGC^-(D), \partial^-\right)}{V = 0} .$$

The *collapsed filtered link Floer homology* $c\mathcal{HFL}^-(L)$ can be seen as the homology of our new complex, in the same manner as in Subsection 8.1.4; moreover, now each level $\mathcal{F}^s c\mathcal{HFL}^-(L)$ has also a structure of an $\mathbb{F}[U]$-module given by $U[p] = [V_i p] = [Vp]$ for every $i = 1, ..., m$ and $[p] \in c\mathcal{HFL}^-(L)$; see [67]. The groups $\mathcal{F}^s c\mathcal{HFL}_d^-(L)$ are still finite dimensional over $\mathbb{F}$ and so we can define the function N as

$$N_L(d,s) = \dim_{\mathbb{F}} \frac{\mathcal{F}^s c\mathcal{HFL}_d^-(L)}{\mathcal{F}^{s-1} c\mathcal{HFL}_d^-(L)} .$$

We expect the function N to be a strong concordance invariant, possibly better than T.

We can compute the function N of the unknot:

$$N_{\bigcirc}(d,s) = \begin{cases} 1 & \text{if } (d,s) = (2t,t), \ \ t \leqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and we know that

$$c\mathcal{HFL}^-(L) \cong_{\mathbb{F}[U]} c\mathcal{HFL}^-(\bigcirc_n) \cong_{\mathbb{F}[U]} \mathbb{F}[U]^{2^{n-1}}$$

as an $\mathbb{F}[U]$-module, where n is the number of component of L.

Since $\dim_{\mathbb{F}} c\mathcal{HFL}_0^-(L)$ is still equal to one, we can define an invariant ν exactly like we did in Subsection 8.2.1 for τ. A version of the ν-invariant has been introduced first by Rasmussen in [78] and he proved that it is a concordance invariant for knots. In [45] Hom and Wu found knots whose ν-invariant gives better lower bound for the slice genus than τ.

Since $H_{*,*}\big(\mathrm{gr}(cGC^-(D), \partial^-)\big)$ is isomorphic to the homology $cGH^-(L) \cong cHFL^-(L)$ of [67, 72] and $N_L(0, \nu(L)) = 1$ for every diagram D of L, we have that $cHFL^-_{0,\nu(L)}(L)$ is non-trivial. Hence, if the homology group $cHFL_0^-(L)$ is non-zero only for one Alexander grading s, we can argue that $\nu(L) = s$. This method can be used to compute the ν-invariant of some links.

We can also define the *(uncollapsed) filtered link Floer homology* as the homology of the complex $\left(GC^-(D), \partial^-\right)$, where $GC^-(D)$ is the free $\mathbb{F}[V_1, ..., V_{\mathrm{grd}\,(D)}]$-module over the grid states of D; there are no special O-markings this time.

We see immediately that for every n-component link L is

$$\mathcal{HFL}^-(L) \cong \mathbb{F}[U_1, ..., U_n]$$

with generator in Maslov grading zero, but the filtration will of course depend on L.

8.4.2 Filtered link homology and the τ-set

In this subsection we prove the following theorem.

Theorem 8.4.1 *For an n-component link the τ-set, defined in [67] as -1 times the Alexander gradings of a homogeneous, free generating set of the torsion-free quotient of $cHFL^-(L)$ as an $\mathbb{F}[U]$-module, coincides with the 2^{n-1} (with multiplicity) values of s where the function T is supported.*

Proof. We use the complex cGC^- to show that, for every n-component link L, an integer s gives $T_{d,s}(L) \neq 0$ for some d if and only if s belongs to the τ-set of L. To do this, given $\mathcal{C}$ a freely and finitely generated $\mathbb{F}[U]$-complex, we define two set of integers: $\tau(\mathcal{C})$ and $t(\mathcal{C})$. First we call $\mathcal{B}_\tau(\mathcal{C})$ a homogeneous, free generating set of the torsion-free quotient of $H_{*,*}(\mathrm{gr}(\mathcal{C}))$ as an $\mathbb{F}[U]$-module; then $\tau(\mathcal{C})$ is the set of $s \in \mathbb{Z}$ such that there is a $[p] \in \mathcal{B}_\tau(\mathcal{C})$ with bigrading $(d, -s)$ for some $d \in \mathbb{Z}$. Similarly, $t(\mathcal{C})$ is the set of the integers s such that the inclusion

$$i_s : \mathcal{F}^{s-1}H_*\left(\frac{\mathcal{C}}{U=0}\right) \longrightarrow \mathcal{F}^s H_*\left(\frac{\mathcal{C}}{U=0}\right)$$

is not surjective. Note that the set $\mathcal{B}_\tau(\mathcal{C})$ is not unique, but $\tau(\mathcal{C})$ and $t(\mathcal{C})$ are well-defined.

We say that $s \in \tau(\mathcal{C})$ has multiplicity k if there are k distinct elements in $\mathcal{B}_\tau(\mathcal{C})$ with bigrading $(*, -s)$, while a number $u \in t(\mathcal{C})$ has multiplicity k if Coker i_u has dimension k as an $\mathbb{F}$-vector space.

Clearly, if D is a grid diagram of L, $t\big(cGC^-(D)\big)$ is the set of the values of s where the function T_L is supported; moreover, we already remarked that $H_{*,*}\left(\mathrm{gr}\left(cGC^-(D)\right)\right)$ is isomorphic to $cHFL^-(L)$ and then $\tau\big(cGC^-(D)\big)$ is the τ-set of L. Hence our goal is to prove that $t\big(cGC^-(D)\big) = \tau\big(cGC^-(D)\big)$.

Consider the complex

$$\mathcal{C} = \frac{cGC^-(D)}{V_1 = ... = V_m = U},$$

where $m = \mathrm{grd}(D) - n$ and U is the variable associated to the special O-markings. Then we define $\overline{\mathcal{C}}$ as the complex $\mathcal{C}[\![1-n-m, -m]\!]$.

We introduce a new complex $\mathcal{C}' = \mathcal{C} \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ and we define a $\mathbb{Z} \oplus \mathbb{Z}$-filtration on $\mathcal{C}'$ in the following way: $\mathcal{F}^{x,*}\mathcal{C}' = U^{-x}\mathcal{C}$ for every $x \in \mathbb{Z}$, $\mathcal{F}^{*,y}\mathcal{C}' = \mathcal{F}^y\mathcal{C}' = \big\{p \in \mathcal{C}' \mid A(p) \leqslant y\big\}$ for every $y \in \mathbb{Z}$ and $\mathcal{F}^{x,y}\mathcal{C}' = \mathcal{F}^{x,*}\mathcal{C}' \cap \mathcal{F}^{*,y}\mathcal{C}' = U^{-x}\mathcal{F}^{y-x}\mathcal{C}$ for every $x, y \in \mathbb{Z}$. The first step is to prove that $\tau(\mathcal{C}) = t(\overline{\mathcal{C}})$.

We have that

$$\frac{\mathcal{C}}{U=0} \cong \widetilde{GC}(D) \cong \frac{\mathcal{F}^{0,*}\mathcal{C}'}{\mathcal{F}^{-1,*}\mathcal{C}'};$$

moreover, for every integer s we claim that

$$\mathcal{F}^s\widetilde{GC}(D) \cong \frac{\mathcal{F}^{0,s}\mathcal{C}'}{\mathcal{F}^{-1,s}\mathcal{C}'}.$$

Since, from [67], it is $\mathcal{F}^{x,y}\mathcal{C}' \cong \mathcal{F}^{y,x}\overline{\mathcal{C}'}$ for every $x, y \in \mathbb{Z}$, then we have that

$$\mathcal{F}^s \frac{\overline{\mathcal{C}}}{U=0} \cong \mathcal{F}^s \widetilde{GC}(D)[\![1-n-m,-m]\!] \cong \frac{\mathcal{F}^{s,0}\mathcal{C}'}{\mathcal{F}^{s,-1}\mathcal{C}'} \,. \tag{8.10}$$

Using this identification we obtain that $t(\overline{\mathcal{C}})$ coincides with set of $s \in \mathbb{Z}$ such that the map

$$H_*\left(\frac{\mathcal{F}^{s-1,0}\mathcal{C}'}{\mathcal{F}^{s-1,-1}\mathcal{C}'}\right) \hookrightarrow H_*\left(\frac{\mathcal{F}^{s,0}\mathcal{C}'}{\mathcal{F}^{s,-1}\mathcal{C}'}\right) \tag{8.11}$$

is not surjective.

Now we consider the complex $\mathrm{gr}(\mathcal{C})$, which is equal to $\bigoplus_{t\in\mathbb{Z}} \frac{\mathcal{F}^{0,t}\mathcal{C}'}{\mathcal{F}^{0,t-1}\mathcal{C}'}$. We have that the map

$$U^t : \frac{\mathcal{F}^{0,t}\mathcal{C}'}{\mathcal{F}^{0,t-1}\mathcal{C}'} \longrightarrow \frac{\mathcal{F}^{-t,0}\mathcal{C}'}{\mathcal{F}^{-t,-1}\mathcal{C}'}$$

is an isomorphism and $\frac{\mathcal{F}^{-t,0}\mathcal{C}'}{\mathcal{F}^{-t,-1}\mathcal{C}'}$ is a subspace of $\frac{\mathcal{F}^{*,0}\mathcal{C}'}{\mathcal{F}^{*,-1}\mathcal{C}'}$ for every integer t. From a result in [67] the latter filtered complex is isomorphic to $\frac{\mathrm{gr}(\mathcal{C})}{U=1}$, but it is also isomorphic to $\widetilde{GC}(D)[\![1-n-m,-m]\!]$ for Equation (8.10). In this way we can define a surjective map

$$\Psi : \mathrm{gr}(\mathcal{C}) \longrightarrow \frac{\mathrm{gr}(\mathcal{C})}{U=1}$$

and it is easy to see that $\Psi\left(\mathcal{B}_\tau(\mathcal{C})\right)$ is still a homogeneous, free generating set of the homology; furthermore, if $[p]$ is a torsion element in $H_{*,*}(\mathrm{gr}(\mathcal{C}))$ then $[\Psi(p)] = [0]$. This means that $\tau(\mathcal{C})$ is the set of $-t \in \mathbb{Z}$ such that the map

$$H_*\left(\frac{\mathcal{F}^{-t-1,0}\mathcal{C}'}{\mathcal{F}^{-t-1,-1}\mathcal{C}'}\right) \hookrightarrow H_*\left(\frac{\mathcal{F}^{-t,0}\mathcal{C}'}{\mathcal{F}^{-t,-1}\mathcal{C}'}\right) \tag{8.12}$$

is not surjective.

If we change $-t$ with s in Equation (8.12) then we immediately see that it coincides with Equation (8.11) and so $\tau(\mathcal{C}) = t(\overline{\mathcal{C}})$. Moreover, we can say that an integer in $\tau(\mathcal{C})$ has multiplicity k if and only if it has multiplicity k in $t(\overline{\mathcal{C}})$. Finally, since the map U^t drops the Maslov grading by $2t$, we have that if there is a $[p] \in \mathcal{B}_\tau(\mathcal{C})$ with bigrading (d,s) then there is a generator of $\widetilde{\mathcal{GH}}(D)[\![1-n-m,-m]\!]$ with grading and minimal level $(d-2s,-s)$.

The second and final step is to show that the previous claim implies $t\big(cGC^-(D)\big) = \tau\big(cGC^-(D)\big)$. From Lemma 8.2.4 we have that the filtered quasi-isomorphisms $\mathcal{C} \cong cGC^-(D) \otimes W^{\otimes m}$ and $\overline{\mathcal{C}} \cong cGC^-(D) \otimes (W^*)^{\otimes m}$, where W is the two dimensional $\mathbb{F}$-vector space with generators in grading and minimal level $(d,s) = (0,0)$ and $(d,s) = (-1,-1)$. Thus $t\big(cGC^-(D)\big)$ and $\tau\big(cGC^-(D)\big)$ are completely determined by $\tau(\mathcal{C})$ and $t(\overline{\mathcal{C}})$, so this means that they coincide and the proof is complete. Obviously, the conclusions about multiplicities and Maslov shifts are still true. □

From [67] we know that there is only one element $[p] \in \mathcal{B}_\tau\left(cGC^-(D)\right)$ with bigrading $(-2\tau_1, -\tau_1)$ and only another one $[q]$ with bigrading $(-2\tau_2 + 1 - n, -\tau_2)$. Then the proof of Theorem 8.4.1 implies that there are two non-zero elements in $\widehat{\mathcal{HFL}}(L)$ in grading and minimal level $(0,\tau_1)$ and $(1-n,\tau_2)$. Since, from the definition of τ and τ^*, we also know that there are only two generators of $\widehat{\mathcal{HFL}}(L)$ in Maslov grading zero and $1-n$; we have the following corollary.

Corollary 8.4.2 *Take a grid diagram D of an n-component link L and consider a set $\mathcal{B}_\tau(cGC^-(D))$. If $[p],[q] \in \mathcal{B}_\tau(cGC^-(D))$ are such that $[p]$ is in bigrading $(-2\tau_1, -\tau_1)$ and $[q]$ is in bigrading $(-2\tau_2 + 1 - n, -\tau_2)$ then $\tau(L) = \tau_1$ and $\tau^*(L) = \tau_2$.*

8.5 Applications

8.5.1 Computation for some specific links

In general it is hard to say when a sum of grid states is a generator of the homology, but the following lemma provides an example where we have useful information.

Lemma 8.5.1 *Suppose L is an n-component link with grid diagram D and $x \in S(D)$ as in Figure 8.13. Then $[x]$ is always the generator of $\widehat{\mathcal{HFL}}_0(L)$ and $c\mathcal{HFL}_0^-(L)$. Furthermore, $\tau(L) = \nu(L) = A(x)$.*

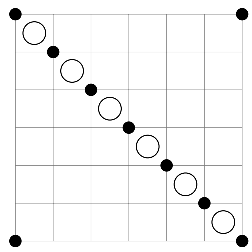


Figure 8.13: We denote with x the grid state in the picture.

Proof. We show that $M(x) = 0$, $\widehat{\partial} x = \partial^- x = 0$ and that for every other grid state y of D it is $M(y) \leqslant 0$.

i) The fact that $M(x) = 0$ is trivial.

ii) For every $y \in S(D)$ there are always 2 rectangles in $\mathrm{Rect}^\circ(x, y)$ and they contain no O, so they cancel when we compute the differential.

iii) We prove by induction on $\mathrm{grd}(D)$ that $M(y) \leqslant 0$ for every $y \in S(D)$.

If $\mathrm{grd}(D) = 1$ then x is the only grid state.

If $\mathrm{grd}(D) = 2$ then there are only x and y and it is $M(y) = -1$.

Suppose the claim is true for the diagrams with dimension equal or smaller than α and let $\mathrm{grd}(D) = \alpha + 1$. We denote with $I(D) \subset S(D)$ the subset of grid states that contain the point $(0, \alpha)$ as in Figure 8.14. Every $y \in I(D)$ is the extension of a grid state y' of the diagram D' obtained by removing the first column and the last row from D. By the inductive hypothesis we have $M(y) = -1 + M(y') \leqslant -1$.

Now it easy to see that every other $z \in S(D)$ is obtained by a rectangle move from a $y \in I(D)$. Then, if r is the rectangle, we have

$$M(z) - M(y) = 1 - 2 \cdot \left| r \cap \mathbb{O} \right| + 2 \cdot \left| \mathrm{Int}(r) \cap y \right|,$$

but $\left|\mathrm{Int}(r) \cap y\right| \leqslant \min\{\pi_1(r), \pi_2(r)\} = \left| r \cap \mathbb{O} \right|$ where $\pi_i(r)$ is the lenght of the edges of r. Hence $M(z) \leqslant 0$.

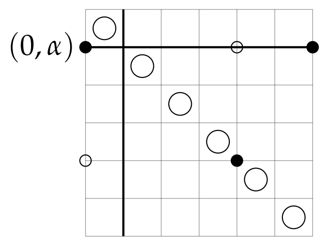


Figure 8.14: The state $y \in I(D)$ is marked with the black circles.

□

In Lemma 8.5.1 we used that the grid diagram D has all the O-markings aligned on a diagonal. It is easy to see that if a link admits such a diagram then it is positive. On the other hand, it seems difficult for the converse to be true.

8.5.2 Torus links

We compute the τ-invariant of every torus link. Consider the grid diagram $D_{q,p}$ in Figure 8.15, representing the torus link $T_{q,p}$ with $q \leqslant p$ and all the components oriented in the same direction. From Lemma 8.5.1 we know that $[x]$ is the only generator of $\widehat{\mathcal{HFL}}_0(T_{q,p})$.

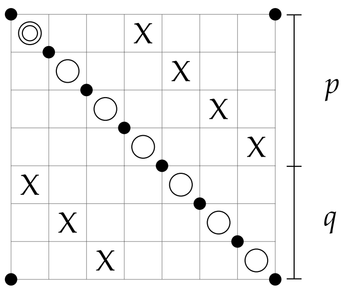


Figure 8.15: x is the grid state in the picture.

If we denote with n the number of components of $T_{q,p}$ then a simple computation gives

$$\begin{aligned} A(x) &= \frac{1}{2}\left(M(x) - M_{\mathbb{X}}(x) - \mathrm{grd}(D_{q,p}) + n\right) = \frac{1}{2}(-M_{\mathbb{X}}(x) - p - q + n) = \\ &= \frac{1}{2}\left[2\sum_{i=1}^{q-1} i + q(p-q+1) - p - q + n\right] = \frac{1}{2}[q(q-1) + q(p-q+1) - p - q + n] = \\ &= \frac{1}{2}(pq - p - q + n) = \frac{(p-1)(q-1) - 1 + n}{2}. \end{aligned}$$

Now we use Lemma 8.5.1 again and obtain that

$$\tau(T_{q,p}) = \frac{(p-1)(q-1) - 1 + n}{2}.$$

Using the lower bound of Equation (8.9) gives a different way to compute the slice genus of a torus link with respect to what we did in [8]:

$$g_4(T_{q,p}) \geqslant \frac{(p-1)(q-1)}{2} - \frac{1-n}{2} + 1 - n = \frac{(p-1)(q-1)+1-n}{2} .$$

Since the Seifert algorithm applied to the standard diagram of $T_{q,p}$ gives the opposite inequality, we conclude that

$$g_4(T_{q,p}) = \frac{(p-1)(q-1)+1-n}{2} \quad \text{for any } q \leqslant p .$$

8.5.3 Applications to Legendrian invariants

We equip S^3 with its unique tight contact structure ξ_{st}, whose definition can be found in Section 3.4. We remark that, if $\mathcal{D} = \mathcal{D}_1 \cup ... \cup \mathcal{D}_n$ is a front projection of the Legendrian link $\mathcal{L}$ in the standard contact 3-sphere, the Thurston-Bennequin and rotation number of $\mathcal{L}$ are given by

$$\text{tb}(\mathcal{L}) = \sum_{i=1}^{n} \text{tb}_i(\mathcal{L}) \quad \text{and} \quad \text{rot}(\mathcal{L}) = \sum_{i=1}^{n} \text{rot}_i(\mathcal{L})$$

where

$$\text{tb}_i(\mathcal{L}) = \text{wr}(\mathcal{D}_i) + \text{lk}\,(\mathcal{D}_i, \mathcal{D} \setminus \mathcal{D}_i) - \frac{1}{2}|\text{cusps in } \mathcal{D}_i|$$

and

$$\text{rot}_i(\mathcal{L}) = \frac{1}{2}\Big(|\text{down-ward cusps in } \mathcal{D}_i| - |\text{up-ward cusps in } \mathcal{D}_i|\Big) .$$

More information can be found in Section 3.2.

Proposition 8.5.2 *Consider a Legendrian n-component link $\mathcal{L}$ of link type L in S^3 equipped with the standard contact structure. Then the following inequality holds:*

$$\text{tb}(\mathcal{L}) + |\,\text{rot}(\mathcal{L})| \leqslant 2\tau(L) - n . \tag{8.13}$$

Proof. If $\mathcal{L}$ is a Legendrian link then, from [67], we know that $\mathcal{L}$ can be represented by a grid diagram D of the link L^* (the mirror of L). This diagram D is such that

$$\frac{\text{tb}_i(\mathcal{L}) - \text{rot}_i(\mathcal{L}) + 1}{2} = A_i(x^+) \quad \frac{\text{tb}_i(\mathcal{L}) + \text{rot}_i(\mathcal{L}) + 1}{2} = A_i(x^-)$$

$$\text{tb}(\mathcal{L}) - \text{rot}(\mathcal{L}) + 1 = M(x^+) \quad \text{tb}(\mathcal{L}) + \text{rot}(\mathcal{L}) + 1 = M(x^-) ,$$

where $x^{\pm}$ are the grid states in D obtained by taking a point in the northeast (southwest for x^-) corner of every square decorated with an $X \in \mathbb{X}$. Moreover, A_i is defined as follows:

$$A_i(x) = \mathcal{J}\left(x - \frac{1}{2}(\mathbb{X} + \mathbb{O}), (\mathbb{X}_i - \mathbb{O}_i)\right) - \frac{\text{grd}(D)_i - 1}{2} \quad \text{for any } x \in S(D)$$

with $\mathbb{O}_i \subset \mathbb{O}$ and $\mathbb{X}_i \subset \mathbb{X}$ the markings on the i-th component of L^* and $\text{grd}(D)_i$ the number of elements in $\mathbb{O}_i$.

In [67] is proved that $x^{\pm}$ represent non-torsion elements in the homology group $cHFL^-(L^*)$; in fact these classes are the Legendrian grid invariants $\lambda^{\pm}(\mathcal{L})$. Since $A(x) =$

$\sum_{i=1}^{n} A_i(x)$ for every grid state x, we have that $M\left(x^{\pm}\right) = 2A\left(x^{\pm}\right) + 1 - n$. Therefore, Corollary 8.4.2 implies that $A\left(x^{\pm}\right) \leqslant -\tau^*(L^*)$. Combining the latter claim with Corollary 8.2.6, that gives $\tau^*(L^*) = -\tau(L)$, we have

$$\frac{\mathrm{tb}(\mathcal{L}) \mp \mathrm{rot}(\mathcal{L}) + n}{2} = A(x^{\pm}) \leqslant \tau(L)$$

that is precisely Equation (8.13). □

From Equation (8.13), together with Equation (8.9), we obtain the following lower bound for the slice genus:

$$\mathrm{tb}(\mathcal{L}) + |\operatorname{rot}(\mathcal{L})| \leqslant 2g_4(L) + n - 2 \leqslant 2u(L) - n\,, \tag{8.14}$$

where $u(L)$ is the unlinking number of L; see [55].

This bound is sharp for positive torus links, but here we show that there are other links for which this happens.

In Figure 8.16 we have a front projection $\mathcal{D}$ of a Legendrian two component link $\mathcal{L}$. The link type of $\mathcal{L}$ is the link $L9^n_{19}$. A simple computation gives $\mathrm{tb}(\mathcal{L}) = 6$ and $\mathrm{rot}(\mathcal{L}) = 0$, therefore Equation (8.14) says that $g_4(L9^n_{19}) \geqslant 3$. Since it is easy to see that the link represented by $\mathcal{D}$ can be unlinked by changing the four crossings highlighted in the picture, we have $g_4(L9^n_{19}) \leqslant u(L9^n_{19}) - 1 \leqslant 3$ and then we conclude that $g_4(L9^n_{19}) = 3$.

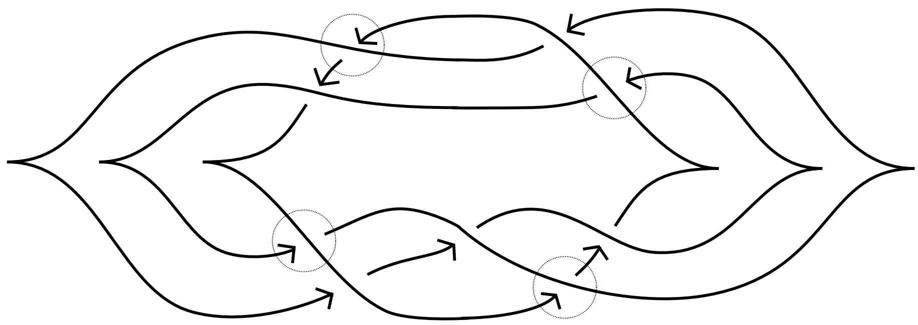


Figure 8.16: A diagram of the link $L9^n_{19}$.

From Equation (8.13) we also have the following upper bound for the maximal Thurston-Bennequin number.

Proposition 8.5.3 *For every n-component link L we have*

$$TB(L) \leqslant 2\tau(L) - n\,.$$

Furthermore, if L is a quasi-alternating link then we have that

$$TB(L) \leqslant -1 - \sigma(L)\,.$$

Although this bound is much less powerful than the Kauffmann or the HOMFLY polynomial, we can still get some interesting conclusions.

Consider a Legendrian link $\mathcal{L}$ such that each component $\mathcal{L}_i$ is algebraically unlinked. Then $\mathrm{tb}_i(\mathcal{L}) = \mathrm{tb}(\mathcal{L}_i)$ and so $\mathrm{tb}(\mathcal{L})$ is precisely the sum of the Thurston-Bennequin numbers of its components. For example this happens for the Borromean link B, whose components B_i are three (algebraically unlinked) unknots. It was shown in [61] that there is no

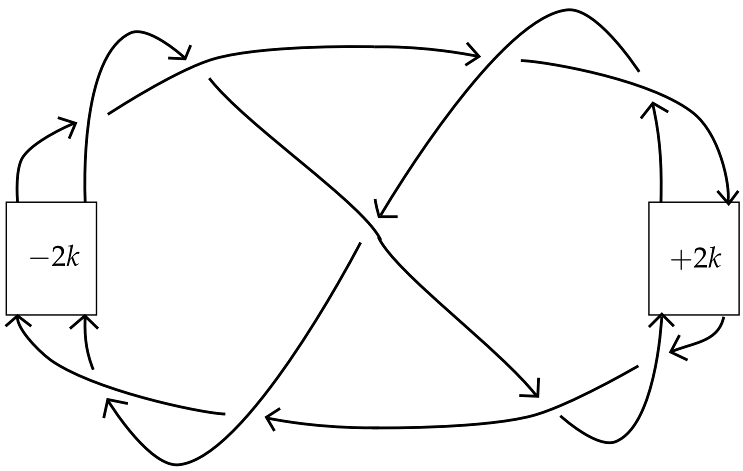


Figure 8.17: A diagram of L^k. For $k = 0$ we have the link $L9^a_{40}$.

Legendrian representation of B, where the Thurston-Bennequin number of each component is -1; in fact we have $\mathrm{TB}(B) = -4$, while $\mathrm{TB}(\bigcirc) = -1$. In particular, this means that the difference between $\mathrm{TB}(B)$ and the sum of $\mathrm{TB}(B_i)$ is -1.

We prove Proposition 8.5.4, where we give a family of 2-components links L^k such that the components of L^k are two unknots with $\mathrm{lk}(L^k_1, L^k_2) = 0$ and the difference between $\mathrm{TB}(L^k)$ and the sum of $\mathrm{TB}(L^k_i)$ is actually arbitrarily small, improving the latter result for B. The links L^k are shown in Figure 8.17.

Proposition 8.5.4 *The links L^k in Figure 8.17 are a family of two component links, whose components L^k_i are unknots with linking number zero, such that $TB(L^k)$ is arbitrarily small.*

Proof. Since for every $k \geqslant 0$ the link L^k is non-split alternating, we can easily compute the signature that is equal to $3 + 2k$. Now we apply Proposition 8.5.3 and we obtain that $\mathrm{TB}(L^k) \leqslant -4 - 2k$. □

As a final obervation, we note that the same argument used by Baldwin, Vela-Vick and Vértesi in [3] gives that the invariant $\mathfrak{L}(L)$ defined in Chapter 6, for Legendrian links in the standard contact 3-sphere, coincides with the grid invariant $\lambda^+(L)$ of [67].

Chapter 9

Quasi-positive links in the 3-sphere

9.1 Maximal self-linking number and the slice genus

A *quasi-positive link* is any link which can be realized as the closure of a d-braid of the form

$$\prod_{i=1}^{b} w_i \sigma_{j_i} w_i^{-1} ,$$

where σ_j for $j = 1, ..., d-1$ are the generators of the d-braids group that we defined in Subsection 3.4.2. Thus quasi-positive links are closures of braids consisting of arbitrary conjugates of positive generators.

It is worth noting that quasi-positive links are equivalent to another, more geometric class of links: the *transverse* $\mathbb{C}$*-links*. These links arise as the transverse intersection of the 3-sphere $S^3 \subset \mathbb{C}^2$, with a complex curve. Transverse $\mathbb{C}$-links include links of isolated curve singularities, but are in fact a much larger class. The fact that quasi-positive links can be realized as transverse $\mathbb{C}$-links is due to Rudolph [80], while the fact that every transverse $\mathbb{C}$-link is quasi-positive is due to Boileau and Orekov [6].

Given a quasi-positive braid $B = (w_1 \sigma_{j_1} w_1^{-1}) \cdot ... \cdot (w_b \sigma_{j_b} w_b^{-1})$, we can associate to B a surface Σ_B as follows. Let us consider the braid B', obtained by removing the σ_{j_i}'s from

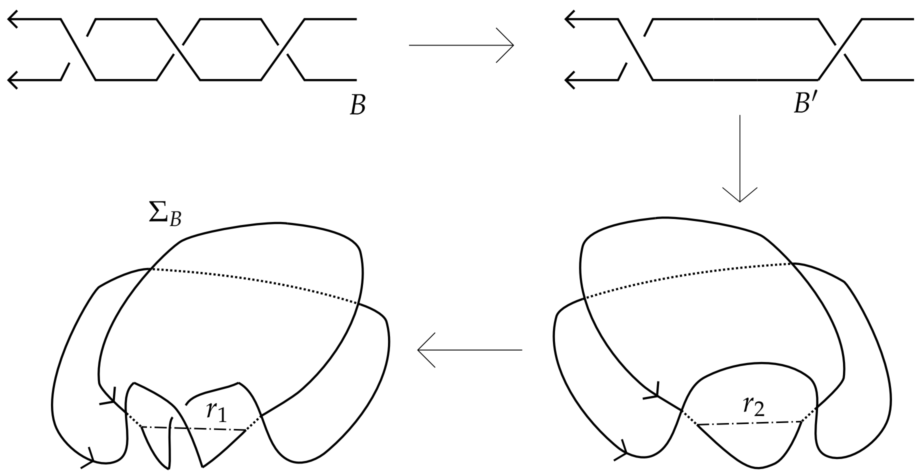


Figure 9.1: The disk Σ_B constructed from the quasi-positive braid B, representing the unknot. The segments r_1 and r_2 are ribbon intersections.

the presentation of B. Then B' is the boundary of d disks with some ribbon interesctions between themselves. If we push these disks in the 4-ball D^4 then the intersections disappear and we obtain a surface which is properly embedded in D^4. At this point, we add b positive bands in correspondence of the σ_{j_i}'s; the result is an oriented and compact surface that is not embedded in S^3, but it is properly embedded in D^4, whose boundary is clearly the closure of the braid B. See Figure 9.1 for an example.

A *connected transverse* $\mathbb{C}$*-link* is a link which is the closure of a quasi-positive braid B such that Σ_B is connected. Then we have the following proposition.

Proposition 9.1.1 *Every connected transverse* $\mathbb{C}$*-link is non-split quasi-positive.*

Proof. Suppose that B is a quasi-positive braid, presented as before, such that Σ_B is connected and its closure L is a split link. Then we can write $L = L_1 \sqcup L_2$ and L_i is the closure of the subbraid B_i for $i = 1, 2$.

Since $\text{lk}(L_1, L_2) = 0$, we have that the arcs corresponding to the σ_{j_i}'s both belong to B_1 or B_2. Therefore, there is no positive band connecting a disk, which appears in the construction of Σ_{B_1}, with the ones in Σ_{B_2}. This implies that Σ_{B_i} is a connected component of Σ_B for $i = 1, 2$ and this is a contradiction. □

Later, in Corollary 9.2.4 we show that this inclusion is strict.

The Thurston-Bennequin inequality stated in Section 3.6 can be improved using the invariant $\tau(L)$, defined from the filtered link Floer homology group $\widehat{\mathcal{HFL}}(L)$ in Chapter 8. More specifically, in Subsection 8.5.3 we prove Equation (8.13):

$$\text{tb}(\mathcal{L}) + |\,\text{rot}(\mathcal{L})| \leqslant 2\tau(L) - n\,,$$

where $\mathcal{L}$ is an n-component Legendrian link in the tight 3-sphere. From this result we obtain the following corollary.

Corollary 9.1.2 *Suppose that* $\mathcal{L}$ *is an* n*-component Legendrian link in* (S^3, ξ_{st}) *with smooth link type* L*. Then we have that*

$$\text{sl}(\mathcal{T}^{\pm}_{\mathcal{L}}) \leqslant \text{tb}(\mathcal{L}) + |\,\text{rot}(\mathcal{L})| \leqslant 2\tau(L) - n \leqslant -\chi(\Sigma)\,, \tag{9.1}$$

where Σ *is an oriented, compact surface, properly embedded in* D^4*, such that* $\partial\Sigma = \mathcal{L}$*.*

Proof. The first two inequalities follow from Proposition 3.3.2 and Equation (8.13). The last one is a consequence of Proposition 8.3.8. □

It is interesting to observe that Equation (9.1) is a version of the adjunction inequality, see [52, 56].

We recall that, from the Thurston-Bennequin inequality, a transverse link in (S^3, ξ_{st}) has bounded self-linking number. This means that we can denote the maximal self-linking number of a link L with $\text{SL}(L)$.

Theorem 9.1.3 *Suppose that* L *is an* n*-component quasi-positive link in* S^3*. Then we have that*

$$SL(L) = b - d \quad \text{and} \quad \tau(L) = \frac{b - d + n}{2}\,,$$

where L *is the closure of a quasi-positive* d*-braid* $B = (w_1\sigma_{j_1}w_1^{-1}) \cdot ... \cdot (w_b\sigma_{j_b}w_b^{-1})$*.*

Proof. As we saw in Subsection 3.4.2, the braid B determines a transverse link $\mathcal{T}$ in (S^3, ξ_{st}) and

$$\mathrm{sl}(\mathcal{T}) = \mathrm{wr}(B) - d = b - d\,.$$

Then from Theorem 6.6.5 and Equation (9.1) we have that

$$\mathrm{sl}(\mathcal{T}) \leqslant \mathrm{SL}(L) \leqslant 2\tau(L) - n \leqslant -\chi(\Sigma_B) = b - d$$

and these inequalities are all equalities. □

When we consider connected surfaces, the inequality in Equation (9.1) becomes

$$\mathrm{sl}(\mathcal{T}) \leqslant 2\tau(L) - n \leqslant 2g_4(L) + n - 2\,, \tag{9.2}$$

for every n-component transverse link $\mathcal{T}$ in (S^3, ξ_{st}), with link type L, and where we recall that $g_4(L)$ is the minimal genus of a compact, oriented surface Σ properly embedded in D^4 and such that $\partial\Sigma = \mathcal{T}$, see Section 1.3. Then we have the following proposition.

Proposition 9.1.4 *Suppose that L is an n-component link, embedded in S^3, which is a connected transverse $\mathbb{C}$-link. Then we have that $\tau(L) = g_4(L) + n - 1$.*

Proof. The claim follows from the same argument in the proof of Theorem 9.1.3 and Equation (9.2). □

An implication of this proposition is the additivity of the slice genus under connected sums.

Corollary 9.1.5 *The slice genus g_4 is additive under connected sums if we restrict to the family of connected transverse $\mathbb{C}$-links.*

Proof. If L_1 and L_2 are two connected transverse $\mathbb{C}$-links then it is easy to see that every connected sum $L = L_1\# L_2$ has the same property. In fact, take two quasi-positive braids, representing L_1 and L_2; then L is obtained by putting the second below the first one and adding a positive band between the components that we want to sum, say the i-th and the j-th ones with $i < j$. This move can be seen as the composition of B with $w\sigma_{j-1}w^{-1}$ for some w and then the resulting braid is still quasi-positive.

Therefore, Proposition 9.1.4 gives that $\tau(L_i) = g_4(L_i) + n_i - 1$, where n_i are the number of components of L_i for $i = 1, 2$. Moreover, what we said before also implies that $\tau(L) = g_4(L) + (n_1 + n_2 - 1) - 1$. At this point, we use that the invariant τ is additive under connected sums of links, see Equation (8.7) in Subsection 8.2.3, and then the proof is complete. □

In Theorem 9.1.3 we showed a way to compute the τ-invariant for quasi-positive links, using braids. It is known that a similar formula holds for the Rasmussen link invariant s, see [76, 83]. Then we can prove Proposition 9.1.6, which says that the two invariants are related to each other in this case.

Proposition 9.1.6 *Take an n-component quasi-positive link L in the 3-sphere. Then the invariant $\tau(L)$ from link Floer homology and the invariant $s(L)$ from Khovanov homology satisfy the relation*

$$s(L) = 2\tau(L) + 1 - n\,.$$

Proof. The invariant $s(L)$ can be computed from a quasi-positive braid B, presented as usual in this section, in the following way:

$$s(L) = b - d + 1\,.$$

This is proved in [76, 83]. Then the statement follows immediately from Theorem 9.1.3. □

9.2 Strongly quasi-positive links

We apply the results obtained in Chapter 8 and in the previous subsection to recompute the invariant tau for strongly quasi-positive links in S^3. A link $L \hookrightarrow S^3$ is *strongly quasi-positive* if it is the boundary of a quasi-positive surface F. Such surfaces are constructed

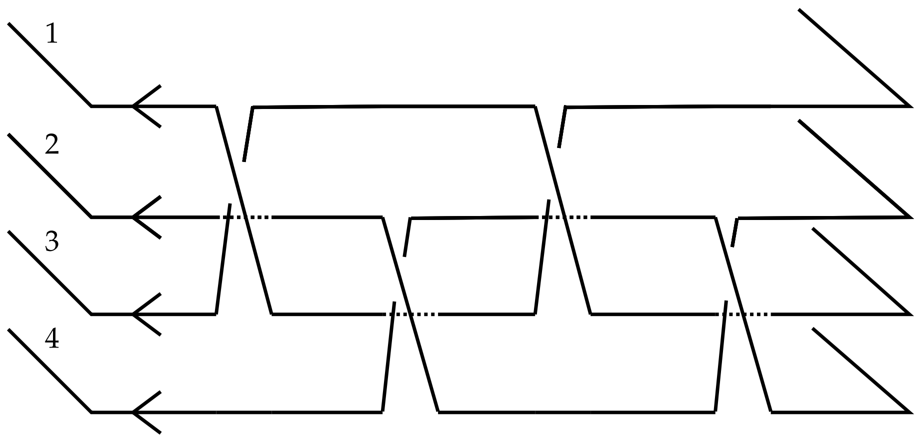


Figure 9.2: The quasi-positive surface determined by $d = 4$ and $\vec{b} = (\sigma_{13}, \sigma_{24}, \sigma_{13}, \sigma_{24})$.

in the following way: take d disjoint parallel, embedded disks, all of them oriented in the same way, and attach b negative bands on them, each one between a pair of distinct disks. This procedure is shown in Figure 9.2. The negative bands cannot be knotted with each other; this means that, up to isotopy, F only depends on the number d and the ordered b-tuple $\vec{b} = (\sigma_{i_1 j_1}, ..., \sigma_{i_b j_b})$, where σ_{ij} denotes that a negative band is put between the i-th and the j-th disk with $i < j$.

Denote with $\sigma_1, ..., \sigma_{d-1}$ the generators of the d-braids group. Then it is easy to see that the boundary of the quasi-positive surface $(1, \sigma_{ij})$ is isotopic to the closure of the d-braid given by

$$\begin{cases} (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1} & \text{if } j \geqslant i+2 \\ \sigma_i & \text{if } j = i+1 \end{cases};$$

see [44] for more details. This immediately implies that strongly quasi-positive links are quasi-positive.

Suppose S^3 is equipped with its unique tight contact structure ξ_{st} and $\mathcal{L} \hookrightarrow (S^3, \xi_{\text{st}})$ is an n-component transverse link with smooth link type L. Then we have the following corollary.

Corollary 9.2.1 *Suppose that $\mathcal{T}$ is an n-component transverse link in (S^3, ξ_{st}) with smooth link type L. Then we have that*

$$\mathrm{sl}(\mathcal{T}) \leqslant 2\tau(L) - n \leqslant \|L\|_T - o(L) \,,$$

where $o(L)$ is the number of disjoint unknots in L.

Proof. The first inequality follows immediately from Equation (9.2). For the second one we apply Theorem 2.1.2; in fact, it is known, see [67], that the group $\widehat{HFL}_{*,*}(L)$ is non-zero in bigrading $(0, \tau(L))$. $\square$

We recall that $\|L\|_T$ is the Thurston norm of L, see Chapter 2. Then we can now prove the following theorem.

Theorem 9.2.2 *Consider a strongly quasi-positive link L with n components in S^3. Then the Thurston-Bennequin inequality is sharp for L, in the sense that there exists a transverse representative $\mathcal{T}$ of L such that*

$$\mathrm{sl}(\mathcal{T}) = 2\tau(L) - n = \|L\|_T - o(L) = -\chi(F)\,,$$

where $o(L)$ is the number of disjoint unknots in L and F is a quasi-positive surface for L. In particular

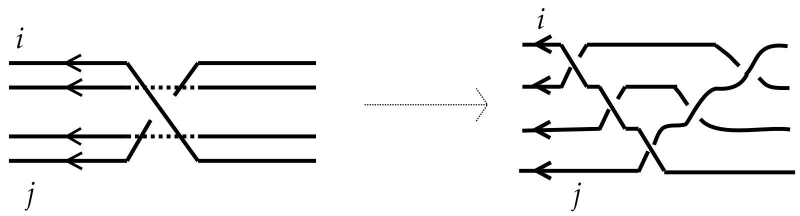


Figure 9.3: Transverse realization of L_1.

$$\tau(L) = \frac{\|L\|_T - o(L) + n}{2} = \frac{b - d + n}{2}\,,$$

where F consists of d disks and b negative bands.

Proof. From Lemma 2.2.5 and Corollary 9.2.1 it is enough to show that, given a quasi-positive surface F whose boundary is L, we can find a transverse link $\mathcal{T}$, with smooth link type L, such that $\mathrm{sl}(\mathcal{T}) = b - d$.

Let us start with the d parallel disks in F and see what happens when we attach the first negative band. The surface we obtain is the quasi-positive surface determined by d and $\vec{b} = (\sigma_{ij})$, where $i < j$ correspond to the disks on which we glue the negative band, and we call L_1 its boundary.

At this point, we choose a transverse representative of L_1 as shown in Figure 9.3. Then $\mathcal{T}$ is defined by iterating this procedure with all the b bands and, since we computed its self-linking number in the previous section, we obtain that

$$\mathrm{sl}(\mathcal{T}) = b - d\,.$$

This proves the claim. $\square$

Since we remarked in the proof of Corollary 9.2.1 that $\widehat{HFL}_{0,\tau(L)}(L) \neq \{0\}$, Theorems 2.1.2 and 9.2.2 tell us that for strongly quasi-positive links

$$\tau(L) = \max\left\{ s \in \mathbb{Z} \mid \widehat{HFL}_{*,s}(L) \neq \{0\} \right\}\,.$$

Furthermore, applying Lemma 2.2.5 to Theorem 9.2.2, we can say more when the link L also bounds a connected quasi-positive surface, that we call *quasi-positive Seifert surface* for L.

Corollary 9.2.3 *If L is a link as in Theorem 9.2.2, which also admits a quasi-positive Seifert surface F', then we have that*

$$\tau(L) = g_3(L) + n - 1 = g(F') + n - 1\,,$$

where $g_3(L)$ is the Seifert genus of L.

This corollary implies that all the quasi-positive Seifert surfaces of a given link have the same genus. Moreover, we observe that not every strongly quasi-positive link admits a quasi-positive Seifert surface. In fact, the 4-component link L in Figure 9.2 is such that $\tau(L) = 2$ from Theorem 9.2.2, but L has Seifert genus zero: to see this it is enough to add a tube between disks 1 and 2 in Figure 9.2. Then we have that

$$2 = \tau(L) \neq g_3(L) + n - 1 = 3$$

and so this would contradict Corollary 9.2.3. This argument also implies the following corollary.

Corollary 9.2.4 *The link L in Figure 9.2 is non-split quasi-positive, but it is not a connected $\mathbb{C}$-transverse link.*

Proof. By construction L is non-split strongly quasi-positive and then it is quasi-positive. If we suppose that L is a connected transverse $\mathbb{C}$-link then we can use Proposition 9.1.4 and we obtain that

$$g_4(L) = \tau(L) + 1 - n = -1$$

which is clearly impossible. □

9.3 The case of positive links

We recall that, from [81], positive links are always strongly quasi-positive. Moreover, if a link is positive non-split then it also admits a quasi-positive Seifert surface. This means that the results in Section 9.2 also hold in this case. Moreover, we state a proposition from [49], which tells us that the maximal Thurston-Bennequin number TB of a positive link can be determined easily.

Proposition 9.3.1 (Kálmán) *Suppose that L is an n-component link in S^3 with a positive diagram D. Then we have that*

$$\mathrm{TB}(L) = c(D) - k(D)\,,$$

where $c(D)$ and $k(D)$ are the numbers of crossings in D and of circles in the oriented resolution of D.

We want to prove the following corollary.

Corollary 9.3.2 *Suppose that L is a link as in Proposition 9.3.1 with split components $L_1, ..., L_r$. Then we have that*

$$\mathrm{TB}(L) = 2\tau(L) - n = 2g_3(L) + n - 2r\,.$$

When a link has a diagram D which is positive, the oriented resolution defined in Subsection 1.2.2 happens to possess some peculiar properties. Let us start with the following lemma.

Lemma 9.3.3 *Suppose that D is a positive diagram for a link L and consider $D' \subset D$ a subdiagram, which represents a split component L' of L.*

Then the surface F', obtained from the oriented resolution of D', is contained as subset in F, the one obtained in the same way from D.

Proof. The surface F' is completely determined by $\mathcal{K}'$, the collection of circles in the oriented resolution of D', and the crossings in D'. Denote with $\mathcal{K}$ the circles in the oriented resolution of D. If $C \in \mathcal{K}'$ then $C \in \mathcal{K}$; this is because $\mathrm{lk}(L', L \setminus L') = 0$, since L' is a split component of L. Therefore, there is no crossing between D' and $D \setminus D'$, because D is positive, and then neither between $\mathcal{K}'$ and $\mathcal{K} \setminus \mathcal{K}'$. Moreover, the crossings in D' also appear in D because D' is a subdiagram of D.

It follows that $F' \subset F$ and $F \setminus F'$ is precisely the surface obtained from the oriented resolution of $D \setminus D'$. This appears clearly from Figure 1.7. □

In order to prove Proposition 9.3.1 we need another lemma.

Lemma 9.3.4 *Suppose that D is a positive diagram for a link $L = L_1 \sqcup ... \sqcup L_r$, where $\{L_i\}_{i=1}^r$ are the split components of L.*

Then the surface F, obtained from the oriented resolution of D, can be isotoped into $F_1 \sqcup ... \sqcup F_r$; where each F_i is a Seifert surface for L_i. Furthermore, F_i is obtained from the oriented resolution of $D_i \subset D$, the subdiagram representing L_i, for every $i = 1, ..., r$.

Proof. Let us work on the case when L is non-split first. The surface F is compact and oriented, but now it is also connected; in fact, otherwise we could write $D = D^1 \cup D^2$ where there is no crossing between D^1 and D^2. This implies that, if $D \subset \{z = 0\}$ in $\mathbb{R}^3$, we could push D^2 up in the plane $\{z = 1\}$ and then move it away from D^1. This determines an isotopy of S^3 and then L would be split, which is a contradiction.

Now, for the general case, we observe that, from Lemma 9.3.3 and the non-split case we did before, the surface F possesses as connected components the surfaces $\{F_i\}_{i=1}^r$ each one being a Seifert surface for L_i; moreover, the F_i's are obtained from the oriented resolution of the subdiagrams $D_i \subset D$ that represent L_i for $i = 1, ..., r$.

We want to isotope F in a way that it becomes the disjoint union of $F_1, ..., F_r$. To do this we consider the planar subspaces $\mathcal{D}$ and $\{\mathcal{D}_i\}_{i=1}^r$ of $\mathbb{R}^2$, associated to D and $\{D_i\}_{i=1}^r$ respectively as in Subsection 1.2.2, and we call F_i an innermost component of F if only one component of $\mathbb{R}^2 \setminus \mathcal{D}_i$ intersects F. Hence, we reason in the same way as before and we push F_i slightly up from the plane and move it away from $F \setminus F_i$. We repeat this procedure for all the components of F and at the end the surface will be as we wanted. □

We are now ready to prove Corollary 9.3.2.

Proof of Proposition 9.3.1 and Corollary 9.3.2. We suppose first that L is a non-split link. Then from Corollaries 9.1.2 and 9.2.1 we know that each Legendrian representative of L satisfies the following inequality:

$$\mathrm{tb}(\mathcal{L}) \leqslant \mathrm{TB}(L) \leqslant 2\tau(L) - n = 2g_3(L) + n - 2 \leqslant -\chi(F) = c(D) - k(D)\,; \qquad (9.3)$$

where F is the surface obtained from the oriented resolution of D, which is a Seifert surface for Lemma 9.3.4. The first equality follows from the fact that a non-split positive link admits a quasi-positive Seifert surface and Corollary 9.2.3.

We define the Legendrian link $\mathcal{L}$ in the following way. Let us consider the planar subspace $\mathcal{D}$ of $\mathbb{R}^2$, associated to D, and the set $\mathcal{K}$ consisting of the circles in the oriented resolution. We isotope the circles in $\mathcal{K}$, starting from the innermost ones, until they have only two vertical tangency points, one on the left and the other one on the right of the circle. At this point, we isotope $\mathcal{D}$ in a way that the lines between the circles in $\mathcal{K}$ appear

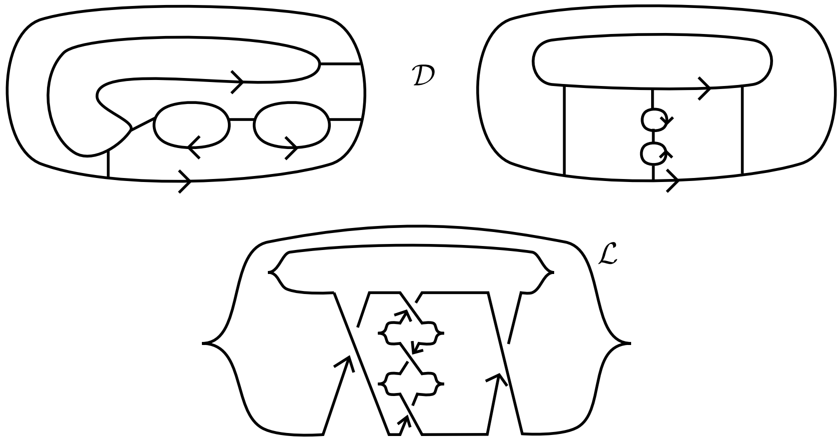


Figure 9.4: Legendrianization of an oriented resolution.

vertical in the plane, see Figure 9.4; this is done by moving the innermost ones first as before, more details can be found in [49].

Then we just change the vertical tangencies into cusps and the vertical lines into crossings; we obtain the front projection of a Legendrian link because D is positive and then all the lines correspond to negative bands. An easy computation gives that $\text{tb}(\mathcal{L}) = c(D) - k(D)$ and $\text{rot}(\mathcal{L}) = 0$, which means that all the inequalities in Equation (9.3) are equalities.

Suppose now that $L = L_1 \sqcup ... \sqcup L_r$. Take the Legendrian links $\mathcal{L}_i$, representing L_i for $i = 1, ..., r$, defined following the same procedure as before and denote with $\mathcal{L}$ the Legendrian link $\mathcal{L}_1 \sqcup ... \sqcup \mathcal{L}_r$. Then we have that

$$\text{tb}(\mathcal{L}) = \sum_{i=1}^{r} \text{tb}(\mathcal{L}_i) = \sum_{i=1}^{r} c(D_i) - k(D_i) = c(D) - k(D)\,, \tag{9.4}$$

where $D_i \subset D$ is the subdiagram representing L_i. Here the final equality follows from Lemma 9.3.4. The Thurston-Bennequin inequality in Corollary 9.2.3 again gives

$$\begin{aligned} \text{tb}(\mathcal{L}) \leqslant TB(L) \leqslant 2\tau(L) - n = \|L\|_T - o(L) \leqslant \sum_{i=1}^{r} (2g_3(L_i) + n_i - 1) \leqslant \\ \leqslant -\sum_{i=1}^{r} \chi(F_i) = c(D) - k(D)\,, \end{aligned} \tag{9.5}$$

where F_i are the Seifert surfaces obtained from the oriented resolution of D_i for $i = 1, ..., r$, the number $o(L)$ tells us how many disjoint unknots there are in L and n_i is the number of components of each L_i. We also used Proposition 2.2.2. Clearly, Equation (9.4) says that all the inequalities in Equation (9.5) are equalities and then the claim follows from the fact that

$$\sum_{i=1}^{r}(2g_3(L_i) + n_i - 1) = 2\sum_{i=1}^{r} g_3(L_i) + n - r = 2g_3(L) + n - r\,,$$

where the final equality holds because the Seifert genus is additive under disjoint unions.

$\square$

The invariant $\tau(L)$ of a positive link can be determined from the oriented resolution of a positive diagram D for L. In fact, Proposition 9.3.1 and its proof immediately imply the following corollary.

Corollary 9.3.5 *Suppose L is a positive n-component link in S^3 with r split components. Then we have that*

$$\tau(L) = \sum_{i=1}^{r} g(F_i) + n - r = \frac{c(D) - k(D) + n}{2},$$

where $\{F_i\}_{i=1}^r$ are the connected components of the surface obtained from the oriented resolution of a positive diagram D for L, while $c(D)$ and $k(D)$ are as in Proposition 9.3.1.

In particular, all the surfaces obtained from the oriented resolution of a positive diagram for a given link have the same genus. Furthermore, Corollary 9.3.5 also gives that a positive link admits a quasi-positive Seifert surface if and only if it is non-split.

Bibliography

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969, pp. ix+128.

[2] K. Baker and J. Etnyre, *Rational linking and contact geometry*, Progr. Math., 296, Birkhäuser/Springer, New York, 2012.

[3] J. Baldwin, D. S. Vela-Vick and V. Vértesi, *On the equivalence of Legendrian and transverse invariants in knot Floer homology*, Geom. Topol., **17** (2013), no. 2, pp. 925-974.

[4] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque, **107-108** (1983), pp. 87-161.

[5] J. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Math. Studies 82, Princeton University Press, Princeton 1975.

[6] M. Boileau and S. Orevkov, *Quasipositivite dune courbe analytique dans une boule pseudo-convexe*, C. R. Acad. Sci. Paris., **332** 2001, pp. 825-830.

[7] D. Calegari, *Foliations and the geometry of 3-manifolds*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.

[8] A. Cavallo, *On the slice genus and some concordance invariants of links*, J. Knot Theory Ramifications, **24** (2015), no. 4, 1550021.

[9] A. Cavallo, *The concordance invariant tau in link grid homology*, to appear on Alg. Geom. Topol., arXiv:1512.08778.

[10] A. Cavallo, *An invariant of Legendrian and transverse links from open book decompositions of contact 3-manifolds*, arXiv:1711.09508.

[11] A. Cavallo, *On loose Legendrian knots in rational homology spheres*, Topology Appl., **235** (2018), pp. 339-345.

[12] A. Cavallo, *On Bennequin type inequalities for links in tight contact 3-manifolds*, arXiv:1801.00614.

[13] V. Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues*, C. R. Acad. Sci. Paris Sér. I Math., **324** (1997), pp. 659-663.

[14] O. Dasbach and B. Mangum, *On McMullen's and other inequalities for the Thurston norm of link complements*, Algebr. Geom. Topol., **1** (2001), pp. 321-347.

[15] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philos. Soc., **136** (2004), no. 3, pp. 583-598.

[16] F. Ding and H. Geiges, *A unique decomposition theorem for tight contact 3-manifolds*, Enseign. Math. (2), **53** (2007), no. 3-4, pp. 333-345.

[17] K. Dymara, *Legendrian knots in overtwisted contact structures on S^3*, Ann. Global Anal. Geom., **19** (2001), no. 3, pp. 293-305.

[18] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math., **98** (1989), no. 3, pp. 623-637.

[19] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Int. J. of Math., **1** (1990), pp. 29-46.

[20] Y. Eliashberg, *On symplectic manifolds with some contact properties*, J. Diff. Geom., **33** (1991), pp. 233-238.

[21] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble), **42** (1992), no. 1-2, pp. 165-192.

[22] Y. Eliashberg, *Classification of contact structures on $\mathbb{R}^3$*, Internat. Math. Res. Notices., (1993), no. 3, pp. 87-91.

[23] Y. Eliashberg, *Legendrian and transversal knots in tight contact 3-manifolds*, Topological methods in modern mathematics, pp. 171-193, Publish or Perish, Houston, TX, 1993.

[24] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, CRM Proc. Lecture Notes, 15, Amer. Math. Soc., Providence, RI, 1998.

[25] Y. Eliashberg and M. Gromov, *Convex symplectic manifolds*, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.

[26] J. Etnyre, *Legendrian and transversal knots. Handbook of knot theory*, Elsevier B. V., Amsterdam, 2005, pp. 105-185.

[27] J. Etnyre, *Lectures on open book decompositions and contact structures*, Clay Math., Proc. (5), Amer. Math. Soc., Providence, RI, 2006, pp. 103-141.

[28] J. Etnyre, *On contact surgery*, Proc. Amer. Math. Soc., **136** (2008), no. 9, pp. 3355-3362.

[29] J. Etnyre, *On knots in overtwisted contact structures*, Quantum Topol., **4** (2013), no. 3, pp. 229-264.

[30] J. Etnyre and K. Honda, *Knots and contact geometry I: torus knots and the figure eight*, Journal of Sympl. Geom., **1** (2001), no. 1, pp. 63-120.

[31] J. Etnyre and K. Honda, *On connected sums and Legendrian knots*, Adv. Math., **179** (2003), no. 1, pp. 59-74.

[32] A. Floer, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math., **41** (1988), no. 4, pp. 393-407.

[33] A. Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math., **41** 1988,no. 6, pp. 775-813.

[34] R. Fox and J. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math., **3** (1966), pp. 257-267.

[35] H. Geiges, *Contact geometry*, Handbook of differential geometry. Vol. II, pp. 315-382, Elsevier/North-Holland, Amsterdam, 2006.

[36] P. Ghiggini, *Strongly fillable contact 3-manifolds without Stein fillings*, Geom. Topol., **9** (2005), pp. 1677-1687.

[37] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv., **66** (1991), no. 4, pp. 637-677.

[38] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math., **141** (2000), no. 3, pp. 615-689.

[39] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II, Higher Ed. Press, Beijing, 2002, pp. 405-414.

[40] R. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2), **148** (1998), no. 2, pp. 619-693.

[41] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999, pp. xvi+558.

[42] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., **82** (1985), no. 2, pp. 307-347.

[43] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[44] M. Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications, **19** (2010), no. 5, pp. 617-629.

[45] J. Hom and Z. Wu, *Four-ball genus bounds and a refinement of the Ozsváth-Szabó τ-invariant*, J. Symplectic Geom., **14** (2016), no. 1, pp. 305-323.

[46] K. Honda, *On the classification of tight contact structures I*, Geom. Topol., **4** (2000), pp. 309-368.

[47] K. Honda, W. Kazez and G. Matić, *On the contact class in Heegaard Floer homology*, J. Differential Geom., **83** (2009), no. 2, pp. 289-311.

[48] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann., **104** (1931), pp. 637-665.

[49] T. Kálmán, *Maximal Thurston-Bennequin number of $+$adequate links*, Proc. Amer. Math. Soc., **136** (2008), no. 8, pp. 2969-2977.

[50] Y. Kanda, *The classification of tight contact structures on the 3-torus*, Comm. in Anal. and Geom., **5** (1997), pp. 413-438.

[51] A. Kawauchi, T. Shibuya and S. Suzuki, *Descriptions on surfaces in four-space. I. Normal forms*, Math. Sem. Notes Kobe Univ., **10** (1982), no. 1, pp. 75-125.

[52] P. Kronheimer and T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters, **1** (1994), pp. 797-808.

[53] J. Levine, *Invariants of knot cobordism*, Invent. Math., **8** (1969), pp. 98-110.

[54] J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv., **44** (1969), pp. 229-244.

[55] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.

[56] P. Lisca and G. Matić, *Stein 4-manifolds with boundary and contact structures*, Topology Appl., **88** (1998), no. 1-2, pp. 55-66.

[57] P. Lisca, P. Ozsváth, A. Stipsicz and Z. Szabó, *Heegaard Floer invariants of Legendrian knots in contact three-manifolds*, J. Eur. Math. Soc., **11** (2009), no. 6, pp. 1307-1363.

[58] C. Manolescu, P. Ozsváth, Z. Szabó and D. Thurston, *On combinatorial link Floer homology*, Geom. Topol., **11** (2007), pp. 2339-2412.

[59] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, NJ, 1965.

[60] J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974, pp. vii+331.

[61] K. Mohnke, *Legendrian links of topological unknots*, AMS, volume 279 of Contemporary Mathematics, 2001, pp. 209-211.

[62] Y. Ni, *Link Floer homology detects the Thurston norm*, Geom. Topol., **13** (2009), no. 5, pp. 2991-3019.

[63] H. Ohta and K. Ono, *Simple singularities and topology of symplectically filling 4-manifold*, Comment. Math. Helv., **74** (1999), pp. 575-590.

[64] S. Orevkov and V. Shevchishin, *Markov Theorem for Transverse Links*, J. Knot Theory Ramifications, **12** (2003), no. 7, pp. 905-913.

[65] B. Ozbagci and A. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Society Mathematical Studies, 13. Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2004. pp. 281.

[66] P. Ozsváth and A. Stipsicz, *Contact surgeries and the transverse invariant in knot Floer homology*, J. Inst. Math. Jussieu, **9** (2010), no. 3, pp. 601-632.

[67] P. Ozsváth, A. Stipsicz and Z. Szabó, *Grid homology for knots and links*, AMS, volume 208 of Mathematical Surveys and Monographs, 2015.

[68] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants of closed three-manifolds*, Ann. of Math. (2), **159** (2004), no. 3, pp. 1159-1245.

[69] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, Adv. Math., **186** (2004), no. 1, pp. 58-116.

[70] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and contact structures*, Duke Math. J., **129** (2005), no. 1, pp. 39-61.

[71] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math., **194** (2005), no. 1, pp. 1-33.

[72] P. Ozsváth and Z. Szabó, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol., **8** (2008), no. 2, pp. 615-692.

[73] P. Ozsváth and Z. Szabó, *Link Floer homology and the Thurston norm*, J. Amer. Math. Soc., **21** (2008), no. 3, pp. 671-709.

[74] J. Pardon, *The link concordance invariant from Lee homology*, Algebr. Geom. Topol., **12** (2012), no. 2, pp. 1081-1098.

[75] T. Perutz, *Hamiltonian handleslides for Heegaard Floer homology*, In Proceedings of Gökova Geometry-Topology Conference 2007, pp. 15-35. Gökova Geometry/Topology Conference (GGT), G̈kova, 2008.

[76] O. Plamenevskaya, *Transverse knots and Khovanov homology*, Math. Res. Lett., **13** (2006), no. 4, pp. 571-586.

[77] O. Plamenevskaya, *A combinatorial description of the Heegaard Floer contact invariant*, Algebr. Geom. Topol., **7** (2007), pp. 1201-1209.

[78] J. Rasmussen, *Floer homology and knot complements*, Thesis (Ph.D.)-Harvard University, 2003, pp. 126.

[79] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, 7. Publish or Perish, Inc., Houston, TX, 1990.

[80] L. Rudolph, *Algebraic functions and closed braids*, Topology, **22** 1983, no. 2, pp. 191-202.

[81] L. Rudolph, *Positive links are strongly quasipositive*, Geom. Topol. Monogr., (2), Geom. Topol. Publ., Coventry, 1999.

[82] S. Sarkar, *Grid diagrams and the Ozsváth-Szabó tau-invariant*, Math. Res. Lett., **18** (2011), no. 6, pp. 1239-1257.

[83] A. Shumakovitch, *Rasmussen invariant, slice-Bennequin inequality, and sliceness of knots*, J. Knot Theory Ramifications, **16** (2007), no. 10, pp. 1403-1412.

[84] J. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc., **35** (1933), no. 1, pp. 88-111.

[85] J. Świątkowski. *On the isotopy of Legendrian knots*, Ann. Global Anal. Geom., **10** (1992), no. 3, pp. 195-207.

[86] W. Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc., **59** (1986), no. 339, pp. i-vi and 99-130.

[87] W. Thurston and H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc., **52** (1975), pp. 345-347.

[88] V. Turaev, *Torsions of 3-manifolds*, Geom. Topol. Monogr. 4, Geom. Topol. Publ., Coventry, 2002.

[89] T. Vogel, *Non-loose unknots, overtwisted discs, and the contact mapping class group of S^3*, arXiv:1612.06557.

[90] F. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, 94. Springer-Verlag, New York-Berlin, 1983. pp. ix+272.

[91] N. Wrinkle, *The Markov theorem for transverse knots*, Thesis (Ph.D.)-Columbia University, 2002, pp. 51.

Index