The geometric genus and Seiberg–Witten invariant of Newton nondegenerate surface singularities

PhD thesis

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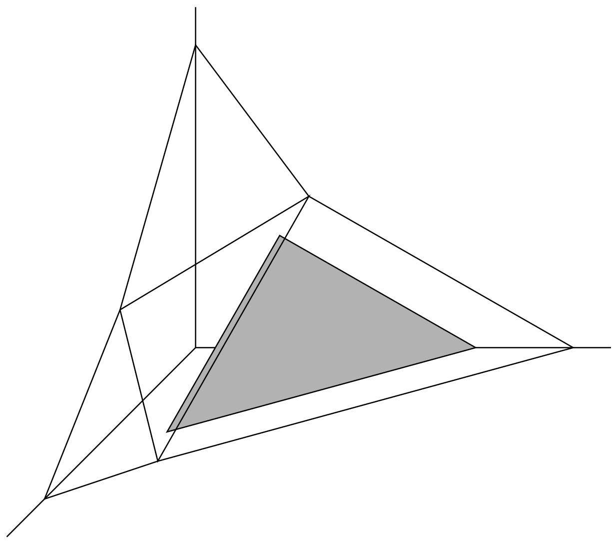

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Abstract

Given a normal surface singularity $(X, 0)$, its link, M, a closed differentiable three dimensional manifold, carries much analytic information. For example, a germ of a normal space is regular if (and only if) its link is the three sphere S^3 [31] (it is even sufficient to assume that $\pi_1(M) = 1$). The geometric genus, p_g, is an analytic invariant of $(X, 0)$ which, in general, cannot be recovered from the link. However, whether $p_g = 0$ can be determined from the link [1]. The same holds for the statement $p_g = 1$, assuming that $(X, 0)$ is Gorenstein [22]. It is an interesting question to ask, whether, under suitable analytic and topological conditions, the geometric genus (or other analytic invariants) can be recovered from the link. The Casson invariant conjecture [50] predicts that p_g can be identified using the Casson invariant in the case when $(X, 0)$ is a complete intersection and M has trivial first homology with integral coefficients (the original statement identifies the signature of a Milnor fiber rather then p_g, but in this case these are equivalent data [23, 70]). The Seiberg–Witten invariant conjecture predicts that the geometric genus of a Gorenstein singularity, whose link has trivial first homology with rational coefficients, can be calculated as a normalized Seiberg–Witten invariant of the link. The first conjecture is still open, but counterexamples have been found for the second one. We prove here the Seiberg–Witten invariant conjecture for hypersurface singularities given by a function with Newton nondegenerate principal part. We provide a theory of computation sequences and how they bound the geometric genus. Newton nondegenerate singularities can be resolved explicitly by Oka's algorithm, and we exploit the combinatorial interplay between the resolution graph and the Newton diagram to show that in each step of the computation sequence constructed, the given bound is sharp. Our method recovers the geometric genus of $(X, 0)$ explicitly from the link, assuming that $(X, 0)$ is indeed Newton nondegenerate with a rational homology sphere link. With some additional information about the Newton diagram, we recover part of the spectrum, as well as the Poincaré series associated with the Newton filtration. Finally, we show that the normalized Seiberg–Witten invariant associated with the canonical spin^c structure on the link coincides with our identification of the geometric genus.

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1 Introduction

This text was written in 2015, in partial fulfillment of the requirements for the degree of doctor of philosophy in mathematics at Central European University in Budapest, under the supervision of Némethi András.

1.1 Content

In section 2 we recall some results on two dimensional singularities and fix notation. These include a formula for the geometric genus in terms of the Poincaré series and a similar formula for the normalized Seiberg–Witten invariant of the link in terms of the zeta function, a general theory of computation sequences, the polynomial part and periodic constant of a power series in one variable, a short review of the spectrum of hypersurface singularities, as well as a result of Saito on part of the spectrum. In the last subsection we give a detailed presentation of our results.

In section 3 we recall the definition of Newton nondegeneracy for a hypersurface singularity, and the construction of its Newton diagram. We recall Oka's algorithm, which provides the graph of a resolution of the singularity from the Newton diagram and discuss conditions of minimality and convenience. Next we recall the Newton filtration and its associated Poincaré series. In the last section we recall a technical classification result which is crucial to the proof in section 7.

In all the following sections, we will assume that $(X, 0)$ is a hypersurface singularity, given by a function with Newton nondegenerate principal part, with a rational homology sphere link. Furthermore, G is the resolution graph produced by Oka's algorithm from the Newton diagram of this function.

In section 4 we fix some notation regarding polygons in two dimensional real affine space, and give a result on counting integral points in such polygons.

In section 5, we construct three computation sequences on G and prove a formula which says that the intersection numbers along these sequences count the integral points under the Newton diagram, or in the positive octant of $\mathbb{R}^3$.

In section 6 we apply the formula from the previous section to prove that the computation sequences constructed calculate the geometric genus, as well as part of the spectrum and the Poincaré series associated with the Newton filtration. In particular, this gives a simple topological identification of the geometric genus for two dimensional Newton nondegenerate hypersurface singularities.

In section 7, we prove that one of the computation sequences constructed in section 5 calculates the normalized Seiberg–Witten invariant for the canonical spin^c structure on the link. As a corollary, we prove the Seiberg–Witten invariant conjecture for $(X, 0)$.

1.2 Acknowledgements

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1.3 Notation

The *content* of an integral vector $a \in \mathbb{Z}^N$ is the greatest common divisor of its coordinates. A *primitive* vector is a vector whose content is 1. If $p, q \in \mathbb{Z}^N$, then we say that the segment $[p, q]$ is primitive if $q - p$ is a primitive vector. If we consider $\mathbb{Z}^N$ as an affine space, and $\ell : \mathbb{Z}^N \to \mathbb{Z}$ is an affine function, then its *content* is the index of its image as a coset in $\mathbb{Z}$. Equivalently, the content c of ℓ is the largest $c \in \mathbb{Z}$ for which there exists an affine function $\tilde{\ell} : \mathbb{Z}^n \to \mathbb{Z}$ and a constant $b \in \mathbb{Z}$ so that $\ell = c\tilde{\ell} + b$. An affine function is *primitive* if its content is 1.

2 General theory and statement of results

In this section we will recall some facts about singularities and fix some notation. We will always assume that $(X, 0)$ is a germ of a normal complex surface singularity, embedded in some $(\mathbb{C}^N, 0)$. Furthermore, when choosing a representative X of the germ $(X, 0)$, we assume X to be a contractible Stein space given as the intersection of a closed analytic set and a suitably small ball around the origin, and that X is smooth outside the origin.

2.1 The link

In this section we denote by $S_r^{d-1} \subset \mathbb{R}^d$ the sphere with radius r around the origin in $\mathbb{R}^d$, by $B_r^d \subset \mathbb{R}^d$ the ball with radius r and by $\bar{B}_r^d$ its closure. For the definition of *plumbing graphs*, we refer to [48, 31, 47, 60]. Recall that each vertex v of a plumbing graph is labelled by two integers, the *selfintersection number* $-b_v$ and the *genus* g_v. Furthermore, denoting the vertex set of the graph by $\mathcal{V}$, then there is an associated $|\mathcal{V}| \times |\mathcal{V}|$ *intersection matrix* I with $I_{v,v} = -b_v$ and $I_{v,w}$ the number of edges between v and w if $v \neq w$.

2.1.1 Definition. Let $(X, 0)$ be a germ of an isolated surface singularity. Its *link* is the three dimensional manifold $M = X \cap S_r^{2N-1}$ where we assume given some embedding $(X, 0) \to (\mathbb{C}^N, 0)$ and the radius $r > 0$ is sufficiently small. As a differentiable manifold, M does not depend on the embedding $(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$, or r (see e.g. [26]).

The topology (or embedded topology) of a singularity is completely encoded in its link (or the embedding $M \hookrightarrow S_r^{2N-1}$ of the link).

2.1.2 Proposition ([30, 26])**.** *Let $(X, 0)$ be a singularity embedded into $(\mathbb{C}^N, 0)$ for some $N > 0$ and let $r > 0$ be small enough. Then the pair $(\bar{B}_r^{2N}, X \cap \bar{B}_r^{2N})$ is homeomorphic to the cone over the pair (S^{2N-1}, M).* □

2.1.3. In [31], Mumford proved that the germ of a normal two dimensional space is smooth if and only if the link is simply connected. He also showed that the link can always be described by a *plumbing graph*. These graphs were studied by Neumann in [48] where he gave a calculus for determining whether two graphs yield the same manifold. Furthermore, every graph is equivalent to a unique minimal graph which is easily determined from the original graph. A plumbing graph for the link may be obtained from a resolution as described in subsection 2.2.

2.1.4 Proposition (Grauert [11]). *Let M be the three dimensional manifold obtained from the plumbing graph G. Then M is the link of some singularity if and only if G is connected and the associated intersection matrix is negative definite.* □

2.1.5 Proposition (Mumford [31]). *Let M be the three dimensional manifold obtained from the plumbing graph G and assume that the associated intersection matrix is negative definite. Let $g = \sum_{v \in \mathcal{V}} g_v$ be the sum of genera of the vertices of G and define c as the first Betti number of the topological realisation of the graph G, that is, number of independent loops. Then $H_1(M, \mathbb{Z})$ has rank $c + 2g$ and torsion the cokernel of the linear map given by the intersection matrix. In particular, we have $H_1(M, \mathbb{Q}) = 0$ if and only if G is a tree and $g_v = 0$ for all vertices v.* □

2.1.6 Definition. A closed three dimensional manifold M is called a *rational homology sphere* (*integral homology sphere*) if $H_i(M, \mathbb{Q}) \cong H_i(S^3, \mathbb{Q})$ ($H_i(M, \mathbb{Z}) \cong H_i(S^3, \mathbb{Z})$). By Poincaré duality, this is equivalent to $H_1(M, \mathbb{Q}) = 0$ ($H_1(M, \mathbb{Z}) = 0$).

2.2 Resolutions of surface singularities

2.2.1 Definition. Let $(X, 0)$ be a normal isolated singularity. A *resolution* of X is a holomorphic manifold $\tilde{X}$, together with a proper surjective map $\pi : \tilde{X} \to X$ so that $E = \pi^{-1}(0)$ is a divisor in $\tilde{X}$ and the induced map $\tilde{X} \setminus E \to X \setminus \{0\}$ is biholomorphic. We refer to E as the *exceptional divisor* of the resolution π. We say that π is a *good* resolution if $E \subset \tilde{X}$ is a *normal crossing divisor*, that is, a union of smooth submanifolds intersecting transversally, with no triple intersections. We will always assume this condition. Write $E = \cup_{v \in \mathcal{V}} E_v$, where E_v are the irreducible components of E. Denote by g_v the genus of (the normalisation of) the curve E_v and by $-b_v$ the Euler number of the normal bundle of E_v as a submanifold of $\tilde{X}$.

2.2.2 Definition. Let $\pi : (\tilde{X}, E) \to (X, 0)$ be a (good) resolution as above. The *resolution graph* G associated with π is the graph with vertex set $\mathcal{V}$ and $|E_v \cap E_w|$ edges between v and w if $v \neq w$ and no loops. It is decorated with the *selfintersection numbers* $-b_v$ and *genera* g_v for $v \in \mathcal{V}$. We denote by δ_v the *degree* of a vertex G, that is, $\delta_v = \sum_{w \neq v} |E_v \cap E_w|$.

2.2.3 Proposition (Mumford [31]). *Let M be the link of a singularity admitting a resolution with resolution graph G. Then M is the plumbed manifold obtained from the plumbing graph G.* □

2.2.4 Proposition (Zariski's main theorem). *If G is the graph of a resolution of a normal singularity, then G is connected.*

Proof. This follows from the fact that E is a connected variety, see e.g. [15], Corollary 11.4. □

2.2.5. Given an embedding of $(X, 0)$ into some smooth space $(\mathbb{C}^N, 0)$, we may take as a representative for the germ an intersection with a *closed* ball of sufficiently small radius. Then, the resolution $\tilde{X}$ is given as a manifold with boundary and $\partial\tilde{X} = M$. In particular, one can consider the perfect pairing $H_2(\tilde{X},\mathbb{Z}) \otimes H_2(\tilde{X}, M, \mathbb{Z}) \to \mathbb{Z}$ which induces a symmetric form $(\cdot,\cdot) : H_2(\tilde{X},\mathbb{Z})^{\otimes 2} \to \mathbb{Z}$.

The exceptional divisor E is a strong homotopy retract of $\tilde{X}$. In particular, $H_2(\tilde{X},\mathbb{Z}) = \mathbb{Z}\langle E_v | v \in \mathcal{V}\rangle$ and $H_2(\tilde{X}, M, \mathbb{Z}) = \mathrm{Hom}(H_2(\tilde{X},\mathbb{Z}),\mathbb{Z})$ is free. If $v \neq w$, then $(E_v, E_w) = |E_v \cap E_w|$. Further, $E_v^2 = (E_v, E_v)$ is the Euler number of the normal bundle of the submanifold $E_v \subset \tilde{X}$. The intersection form is negative definite, in particular, nondegenerate [31]. This means that the natural map $H_2(\tilde{X},\mathbb{Z}) \to H_2(\tilde{X}, M, \mathbb{Z})$ may be viewed as an inclusion with finite cokernel. In particular, we may view $H_2(\tilde{X}, M, \mathbb{Z})$ as a lattice in $H_2(\tilde{X},\mathbb{Z}) \otimes \mathbb{Q}$, containing $H_2(\tilde{X},\mathbb{Z})$ with finite index.

2.2.6 Definition. Let $L = H_2(\tilde{X},\mathbb{Z}) = \mathbb{Z}\langle E_v|v \in \mathcal{V}\rangle$ and $L' = H_2(\tilde{X}, M, \mathbb{Z}) = \mathrm{Hom}(L,\mathbb{Z})$. The form $(\cdot,\cdot) : L \otimes L \to \mathbb{Z}$ defined above is the *intersection form*. We extend the intersection form to $L_\mathbb{Q} = L \otimes \mathbb{Q}$ and $L_\mathbb{R} = L \otimes \mathbb{R}$ by linearity. Elements of L (or $L_\mathbb{Q}$, $L_\mathbb{R}$) will be referred to as *cycles* with integral (rational, real) coefficients. We set $H = L'/L$. The intersection form is encoded in the *intersection matrix* $I = ((E_v, E_w))_{v,w\in\mathcal{V}}$. This matrix is invertible over $\mathbb{Q}$, and we write $I^{-1} = (I^{-1}_{v,w})$.

2.2.7 Remark. By the above discussion, it is clear that we have an identification $L' = \{l \in L_\mathbb{Q} \,|\, \forall l' \in L : (l, l') \in \mathbb{Z}\}$.

2.2.8 Definition. The *canonical cycle* $K \in L'$ is the unique cycle satisfying the *adjunction equalities* $(K, E_v) = -E_v^2 + 2g_v - 2$. We define the *anticanonical cycle* as $Z_K = -K$. We say that G is *numerically Gorenstein* if $K \in L$.

2.2.9 Remark. (i) The nondegeneracy of the intersection form guarantees the existence of Z_K as a cycle with rational coefficients. By remark 2.2.7 we have $Z_K \in L'$. For hypersurface singularities (more generally, for Gorenstein singularities) we have, in fact, $Z_K \in L$. Indeed, K is numerically equivalent to the divisor defined by any meromorphic differential form on $\tilde{X}$. In the case of a hypersurface singularity (or, more generally, a Gorenstein singularity), there exists a meromorphic 2-form on $\tilde{X}$ whose divisor is exactly K. For details, see e.g. [7, 33].

(ii) This definition of the canonical cycle assumes that all components E_v are smooth. If this is not the case, the correct formula also contains a term counting the "number of nodes and cusps" on E_v, see e.g. [33].

(iii) An isolated singularity $(X, 0)$ is said to be *Gorenstein* if the canonical line bundle $\Omega^2_{X\setminus\{0\}}$ in a punctured neighbourhood around 0 is trivial. Gorenstein singularities are numerically Gorenstein [7, 33] and hypersurfaces (more generally, complete intersections) are Gorenstein [33]. Similarly, $(X, 0)$ is said to be $\mathbb{Q}$-Gorenstein if some tensor power of $\Omega^2_{X\setminus\{0\}}$ is trivial.

2.2.10 Definition. The *dual cycles* $E_v^* \in L'$, $v \in \mathcal{V}$, are defined by the linear equations $(E_v^*, E_w) = -\delta_{v,w}$, where $\delta_{v,w}$ is the Kronecker delta. These exist and are well defined since the intersection matrix I is invertible over $\mathbb{Q}$. In fact, we have $E_v^* = \sum_{w\in\mathcal{V}} -I^{-1}_{v,w} E_w$. It follows that the family $(E_v^*)_{v\in\mathcal{V}}$ is a basis of L'. In particular, we have $E_v^* \in L$ for all $v \in \mathcal{V}$ if and only if M is an integral homology sphere.

2.2.11 Definition. For a cycle $Z = \sum_{v\in\mathcal{V}} m_v E_v \in L$, write $m_v(Z) = m_v$.

2.2.12 Lemma. *The entries $m_w(E_v^*) = -I^{-1}_{v,w}$ are positive.*

Proof. Write $E_v^* = Z_1 - Z_2$, where $m_v(Z_i) \geq 0$ for all v and $i = 1, 2$, and Z_1, Z_2 have disjoint supports (the *support* of a cycle is $\mathrm{supp}(Z) = \{v \in \mathcal{V} \,|\, m_v(Z) \neq 0\}$). Since $(-Z_2, E_v) \leq (Z, E_v) \leq 0$ for all $v \in \mathrm{supp}(Z_2)$, we find $Z_2^2 \geq 0$, hence $Z_2 = 0$ by negative definiteness and so $Z = Z_1$. We must show $\mathrm{supp}(Z_1) = \mathcal{V}$. Since $Z \neq 0$, if there is a $v \in \mathcal{V}\backslash\mathrm{supp}(Z)$, we may assume that there is such a v having a neighbour in $\mathrm{supp}(Z)$. This would give $(Z, E_v) = \sum_{u\in\mathcal{V}_v\cap\mathrm{supp}(Z)} m_u(Z) > 0$ contradicting our assumptions. □

2.3 The topological semigroup

Throughout this subsection we assume given a good resolution $\pi : \tilde{X} \to X$ as described in the previous subsection. We also assume that the link M is a rational homology sphere.

2.3.1 Definition. The *Lipman cone* is the set $\mathcal{S}_{\mathrm{top}} = \{Z \in L \,|\, \forall v \in \mathcal{V} : (Z, E_v) \leq 0\}$. Define also $\mathcal{S}'_{\mathrm{top}} = \{Z \in L' \,|\, \forall v \in \mathcal{V} : (Z, E_v) \leq 0\}$.

2.3.2 Remark. We have $\mathcal{S}'_{\mathrm{top}} = \mathbb{N}\,\langle E_v^* | v \in \mathcal{V}\rangle$ and $\mathcal{S}_{\mathrm{top}} = \mathcal{S}'_{\mathrm{top}} \cap L$.

2.3.3 Proposition. *Let $g \in \mathcal{O}_{X,0}$ and define $Z \in L$ by setting $m_v(Z)$ equal to the divisorial valuation of $\pi^* g$ along $E_v \subset \tilde{X}$. Then $Z \in \mathcal{S}_{\mathrm{top}}$.*

Proof. We have $(g) = \sum_{v\in\mathcal{V}} m_v(Z) E_v + S$ where S is a divisor, none of whose components are supported on E. In particular, we have $(E_v, S) \geq 0$ for all $v \in \mathcal{V}$. Furthermore, (g) is linearly equivalent to 0 in the divisor group, which gives $(E_v, (g)) = 0$ for all v. Thus, $(E_v, Z) = -(E_v, S) \leq 0$. □

2.3.4 Proposition (Artin [1])**.** *The Lipman cone is closed under addition, and therefore makes up a semigroup. Further, for $Z_i = \sum_v m_{v,i} E_v$, $i = 1, 2$, define their* meet *as $Z_1 \wedge Z_2 = \sum_v \min\{m_{v,1}, m_{v,2}\} E_v$. If $Z_1, Z_2 \in \mathcal{S}_{\mathrm{top}}$, then $Z_1 \wedge Z_2 \in \mathcal{S}_{\mathrm{top}}$.*

Proof. The first statement is clear, since $\mathcal{S}_{\mathrm{top}} \subset L$ is given by inequalities, and is therefore given as the integral points in a real convex cone. For the second statement, set $m_v = \min\{m_{v,1}, m_{v,2}\}$. Assuming $Z_1, Z_2 \in \mathcal{S}_{\mathrm{top}}$, and, say, $m_v = m_{v,1}$, we get

$$(Z_1 \wedge Z_2, E_v) = m_{v,1} E_v^2 + \sum_{w\neq v} m_w (E_v, E_w) \leq m_{v,1} E_v^2 + \sum_{w\neq v} m_{w,1} (E_v, E_w) \leq 0.$$

□

2.3.5 Definition. By lemma 2.2.12, the elements in $\mathcal{S}_{\mathrm{top}}$ have positive entries. Therefore, the partially ordered set $\mathcal{S}_{\mathrm{top}}\backslash\{0\}$ has minimal elements. Furthermor, by lemma 2.2.12, the meet $Z_1 \wedge Z_2$ of two elements $Z_1, Z_2 \in \mathcal{S}_{\mathrm{top}} \setminus \{0\}$ is again nonzero. Thus, the set $\mathcal{S}_{\mathrm{top}} \setminus \{0\}$ contains a unique minimal element. We denote this element by Z_{min} and call it *Artin's minimal cycle*, or, the *minimal cycle*. This element is often referred to as the *fundamental cycle*.

2.4 Topological zeta and counting functions

2.4.1. We will make use of the set $\mathbb{Z}[[t^L]] = \big\{\sum_{l\in L} a_l t^l \,\big|\, a_l \in \mathbb{Z}\big\}$. It is a group under addition, and has a partially defined multiplication. More precisely, if $A(t) = \sum a_l t^l$ and $B(t) = \sum b_l t^l$ are elements of $\mathbb{Z}[[t^L]]$, then $A(t)\cdot B(t)$ is defined if the sum $c_l = \sum_{l_1+l_2=l} a_{l_1} b_{l_2}$ is finite for all $l \in L$, in which case we define $A(t)\cdot B(t) = \sum_{l\in L} c_l t^l$. In particular, $\mathbb{Z}[[t^L]]$ is a module over the ring of Laurent polynomials $\mathbb{Z}[t_1^{\pm 1},\dots,t_s^{\pm 1}]$ where $s = |\mathcal{V}|$. A simple exercise also shows that if $A(t) = \sum a_l t^l$ is supported in the Lipman cone (that is, $a_l = 0$ for $l \notin \mathcal{S}_{\mathrm{top}}$) then $A(t)\cdot \sum_{l \ngeq 0} t^l$ is well defined.

In precisely the same way, one obtains the set $\mathbb{Z}[[t^{L'}]] = \big\{\sum_{l\in L'} a_l t^l \,\big|\, a_l \in \mathbb{Z}\big\}$ which naturally contains $\mathbb{Z}[[t^L]]$, and is contained in $\mathbb{Z}[[t_1^{\pm 1/d},\dots,t_s^{\pm 1/d}]]$, where $d = |H|$.

One may modify this definition by introducing coefficients from any ring R, thus obtaining $R[[t^L]]$.

For a discussion of these sets and some rings contained in them, see e.g. [37].

2.4.2 Remark. If $C \subset L_{\mathbb{R}}$ is a strictly convex cone (i.e. contains no nontrival linear space) and $A(t), B(t) \in \mathbb{Z}[[t^{L'}]]$ as above, with $a_{l'} = b_{l'} = 0$ if $l' \notin C$, then $A(t)\cdot B(t)$ is well defined. As is easily seen, the set of such series thus form a local ring, with maximal ideal the set of series $A(t)$ with $a_0 = 0$. In particular, if $l' \in C$, and $l' \neq 0$ then the element $1 - t^{l'}$ is invertible, in fact we have $(1-t^{l'})^{-1} = \sum_{k=0}^{\infty} t^{kl'}$. Since this is independent of the cone C, we will assume this formula without referring to C.

2.4.3 Definition. For $l \in L'$, denote by $[l] \in H = L'/L$ the associated residue class. Denote by $\hat{H} = \mathrm{Hom}(H,\mathbb{C})$ the Pontrjagin dual of the group H. The intersection product induces an isomorphism $\theta : H \to \hat{H}$, $[l] \mapsto e^{2\pi i (l,\cdot)}$. The *equivariant zeta function* associated with the resolution graph G is

$$Z(t) = \prod_{v\in\mathcal{V}} (1 - [E_v^*] t^{E_v^*})^{\delta_v - 2} \in \mathbb{Z}[H][[t^{L'}]]. \tag{2.1}$$

The natural bijection $\mathbb{Z}[H][[t^{L'}]] \leftrightarrow \mathbb{Z}[[t^{L'}]][H]$ induces well defined series $Z_h(t) \in \mathbb{Z}[[t^{L'}]]$ for each $h \in H$ so that $Z(t) = \sum_{h\in H} Z_h(t) h$. It is clear that the series $Z_h(t)$ is supported on the coset of L in L' corresponding to h, that is, the coefficient of $l' \in L'$ in $Z_h(t)$ vanishes if $[l'] \neq h$. In particular, we have $Z_0(t) \in \mathbb{Z}[[t^L]]$, where 0 denotes the trivial element of H. We call $Z_0(t)$ the *zeta function* associated with the graph G. Denote by $z_{l'}$ the coefficients of $Z(t)$, i.e. $Z(t) = \sum_{l'\in L'} z_{l'} t^{l'}$. Thus, we have $Z_0(t) = \sum_{l\in L} z_l t^l$.

The *equivariant counting function* associated with G is the series $Q(t) = \sum_{l'\in L'} q_{l'} [l'] t^{l'} \in \mathbb{Z}[H][[t^{L'}]]$, where $q_{l'} = \sum \{ z_{l'+l} \,|\, l \in L,\, l \ngeq 0\}$. This yields a decomposition $Q(t) = \sum_{h\in H} Q_h(t) h$ where $Q_h \in \mathbb{Z}[[t^{L'}]]$ as above. In particular, $Q_0(t) \in \mathbb{Z}[[t^L]]$. The series $Q_0(t)$ is called the *counting function* associated with G.

2.4.4 Remark. The zeta function is supported on the Lipman cone, that is, writing $Z_0(t) = \sum_{l\in L} z_l t^l$ we have $z_l = 0$ if $l \notin \mathcal{S}_{\mathrm{top}}$.

2.5 The geometric genus

2.5.1 Definition. Let $(X,0)$ be a normal surface singularity. The *geometric genus* of $(X,0)$ is defined as $p_g = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$, where $\tilde{X} \to X$ is a resolution.

The geometric genus of $(X,0)$ is defined in terms of a resolution. Using the fact that any resolution is obtained by blowing up the minimal resolution, as well as Lemma 3.3 from [21], one finds that p_g is independent of the resolution. This fact also follows from the following formula of Laufer:

2.5.2 Proposition (Laufer [21], Theorem 3.4)**.** *We have*

$$p_g = \dim_{\mathbb{C}} \frac{H^0(X\setminus 0, \Omega^2_{X\setminus 0})}{H^0_{L^2}(X\setminus 0, \Omega^2_{X\setminus 0})}, \tag{2.2}$$

where $H^0(X\setminus 0, \Omega^2_{X\setminus 0})$ is the set of germs of holomorphic two forms defined around the origin, and $H^0_{L^2}(X\setminus 0, \Omega^2_{X\setminus 0})$ is the subset of square integrable forms. □

2.5.3. Assume that we have a resolution $\pi : \tilde{X} \to X$ as in 2.2 and take $\omega \in H^0(X\setminus 0, \Omega^2_{X\setminus 0})$. By Laufer [21], ω is square integrable if and only if $\pi^*(\omega)$ extends to a holomorphic form on $\tilde{X}$.

2.5.4 Definition. Assume given a resolution $\pi : \tilde{X} \to X$, with notation as in 2.2. The *divisorial filtration* is a multiindex filtration of $\mathcal{O}_{X,0}$ by ideals, given by

$$\mathcal{F}(l) = \{ f \in \mathcal{O}_{X,0} \,|\, \mathrm{div}(f) \geq l \}, \quad l \in L.$$

For $l \in L$ we set $h_l = \dim_{\mathbb{C}} \mathcal{O}_{X,0}/\mathcal{F}(l)$ and define the *Hilbert series* as $H(t) = \sum_{l\in L} h_l t^l \in \mathbb{Z}[[t^L]]$. The *Poincaré series* is defined as $P(t) = \sum_{l\in L} p_l t^l = -H(t)\prod_{v\in\mathcal{V}}(1-t_v^{-1})$.

2.5.5 Proposition (Némethi [37])**.** *The Poincaré series is supported on the Lipman cone, that is, if $l \notin \mathcal{S}_{\mathrm{top}}$ then $p_l = 0$.* □

2.5.6. The Poincaré series is obtained by a simple formula from the Hilbert series. There are, however, nonzero elements in $\mathbb{Z}[[t^L]]$ whose product with $1 - t_v^{-1}$ is defined and equals zero. This means that, in principle, one can not use this formula to determine H from P. The following proposition guarantees that one may nonetheless determine H from P. The two series therefore provide equivalent data.

2.5.7 Proposition (Némethi [37])**.** *Let H and P be as in definition 2.5.4. Then, for any $l \in L$, we have*

$$h_l = \sum_{\substack{l' \in L \\ l' \not\geq l}} p_{l'}.$$

Equivalently, we have $H(t) = \left(\sum_{l \not\geq 0} t^l\right) \cdot P(t)$. □

2.5.8 Proposition (Némethi [37, 39]). *Assume the notation in 2.2 and 2.3 and let H and P be as in 2.5.4. Then, if $l \in L$ and $(l, E_v) \le (Z_K, E_v)$ for all $v \in \mathcal{V}$, then*

$$h_l = p_g + \frac{(Z_K - l, l)}{2}.$$

$\square$

2.5.9. Combining proposition 2.5.7 and proposition 2.5.8, one finds that the geometric genus can be calculated once the Poincaré series is known. In particular, if one finds a formula for the Poincaré series given in terms of the link M, one automatically obtains a topological identification of p_g. Although this is indeed impossible in general, there are certain cases where the Poincaré series, or just p_g, can be described by topological invariants. As an example, we have the following result:

2.5.10 Proposition (Némethi [39, 37]). *Let $(X, 0)$ be a* splice quitient *singularity [51]. Then the $P(t) = Z_0(t) \in \mathbb{Z}[[t^L]]$. In particular, this holds if $(X, 0)$ is rational, minimally elliptic or weighted homogeneous.*

2.6 The Seiberg–Witten invariants

We will now discuss the *Seiberg–Witten invariants* $\mathbf{sw}^0_M(\sigma) \in \mathbb{Q}$ associated with any three dimensional manifold M with a spin^c structure σ. The definition of these numbers is quite involved and we will only touch the surface of the theory here. For details, see [25] and references therein. There are, however, various identifications of the Seiberg–Witten invariants. In [28], Meng and Taubes proved that in the case $H_1(M, \mathbb{Q}) \neq 0$, the Seiberg–Witten invariants are equivalent to Milnor torsion. Nicolaescu then proved [55] that in the case of a rational homology sphere, the Seiberg–Witten invariants are given by the Casson–Walker invariant and Reidemeister–Turaev torsion . In this case, $\mathbf{sw}^0_M(\sigma)$ is also given as the normalized Euler characteristic of either Ozsváth and Szabó's Heegaard–Floer homology [62], or Némethi's lattice homology associated with σ, see subsection 2.9.

As in 2.2, we use the notation $H = H_1(M, \mathbb{Z})$.

2.6.1. We start with a short review ofi spin^c structures. For more details, see e.g. [54, 40]. For each $n \ge 0$ we have the group $\mathrm{Spin}^c(n)$, along with a U(1) bundle $\mathrm{Spin}^c(n) \to \mathrm{SO(n)}$. This is (for $n \ge 0$) the U(1) bundle corresponding to the nontrivial element in $H^2(\mathrm{SO(n)}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Let X be a CW complex, and let $E \to X$ be a real vector bundle of rank n obtained via a map $\rho : X \to \mathrm{BSO(n)}$. A spin^c *structure* on E is a lifting $X \to \mathrm{BSpin}^c(n)$ of ρ. Since $\ker(\mathrm{Spin}^c(n) \to \mathrm{SO(n)}) = \mathrm{U}(1)$, the difference of two spin^c structures is a U(1) bundle, which is zero if and only the two structures coincide. The set $\mathrm{Spin}^c(E)$ of spin^c structures on E is therefore a torsor over $H^2(X, \mathbb{Z}) = [X, \mathrm{BU}(1)]$, unless it is empty. A spin^c structure on a manifold M is by definition a spin^c structure on its tangent bundle, their set is denoted by $\mathrm{Spin}^c(M)$. Denote the action by $H^2(X, \mathbb{Z}) \times \mathrm{Spin}^c(E) \ni (h, \sigma) \mapsto h\sigma \in \mathrm{Spin}^c(E)$.

The map $\mathrm{U}(n) \to \mathrm{SO(2n)}$ factors through $\mathrm{Spin}^c(2n)$. A complex structure on a vector bundle of even rank therefore induces a spin^c structure. In particular, if E has a complex structure, then $\mathrm{Spin}^c(E) \neq \emptyset$.

Now, assume that M is the boundary of a complex surface $\tilde{X}$. We want to construct the *canonical* spin^c *structure* $\sigma_{\mathrm{can}} \in \mathrm{Spin}(M)$ on M. Note first that since $\tilde{X}$ is a complex manifold, its tangent bundle has a complex structure which induces $\bar{\sigma}_{\mathrm{can}} \in \mathrm{Spin}^c(\tilde{X})$. Now, the tangent bundle of $\tilde{X}$ splits on M as $T\tilde{X}|_M = \mathbb{R} \oplus TM$, where we denote simply by $\mathbb{R}$ the trivial line bundle. Here, the first summand is generated by an outwards pointing vector field. This yields a lift $M \to \mathrm{BSO}(3)$ of the structure map defining $T\tilde{X}|_M$, and this map defines the tangent bundle of M. We therefore have lifts of $M \to \mathrm{BSO}(4)$ to $\mathrm{BSO}(3)$ as well as $\mathrm{BSpin}^c(4)$. Since $\mathrm{Spin}^c(3) = \mathrm{Spin}^c(4) \times_{\mathrm{SO}(4)} \mathrm{SO}(3)$, this defines a lift $M \to \mathrm{BSpin}^c(3)$ of the structure map of the tangent space TM. This is the canonical spin^c structure σ_{can} on M. By the above statements, we get a bijection $H = H^2(M, \mathbb{Z}) \leftrightarrow \mathrm{Spin}^c(M)$ given by $h \leftrightarrow h\,\sigma_{\mathrm{can}}$.

2.6.2. Let M be a compact three dimensional differentiable manifold and choose a spin^c structure σ on M. We will assume throughout that the first Betti number of M is zero, that is, $H_1(M, \mathbb{Q}) = 0$. Choose a Riemannian metric g and a closed two form η on M. Assuming that g and η are chosen sufficiently generic, one obtains a space of *monopoles*, whose signed count we denote by $\mathbf{sw}_M(\sigma, g, \eta)$. Unnormalized Seiberg–Witten invariant This number depends on the choice of g and η. The *Kreck–Stolz invariant* $\mathrm{KS}_M(\sigma, g, \eta) \in \mathbb{Q}$ is another number defined by this data. The *normalized Seiberg–Witten invariants* $\mathbf{sw}^0_M(\sigma)$ are defined as follows:

2.6.3 Proposition (Lim [25]). *The number* $\mathbf{sw}^0_M(\sigma) = \mathbf{sw}_M(\sigma, g, \eta) + \mathrm{KS}_M(\sigma, g, \eta)$ *is independent of the choice of* g *and* η. □

2.6.4 Remark. Lim also obtained results in the case when the first Betti number is greater or equal to 1. We will not discuss these results here, since our results concern rational homology spheres only.

2.6.5. Let M be a rational homology sphere with a spin^c structure σ. Denote by $\lambda(M)$ the *Casson–Walker–Lescop invariant* of M, normalized as in [24]. Denote by

$$\mathcal{T}_{M,\sigma} = \sum_{h \in H} \mathcal{T}_{M,\sigma}(h) h \in \mathbb{Q}[H]$$

the *Reidemeister–Turaev torsion* defined in [68, 69]. The *normalized* (or *modified*) *Reidemeister–Turaev torsion* is defined as

$$\mathcal{T}^0_{M,\sigma} = \sum_{h \in H} \left(\mathcal{T}_{M,\sigma}(h) - \frac{\lambda(M)}{|H|} \right) h \in \mathbb{Q}[H].$$

These invariants are discussed in [40].

2.6.6 Remark. The *Casson*, *Casson–Walker* and *Casson–Walker–Lescop* invariants are successive generalizations. Casson introduced an integral invariant $\lambda_C(M)$ for M an integral homology sphere. For M a rational homology sphere, Walker defined $\lambda_{CW}(M)$ satisfying $\lambda_{CW}(M) = 2\lambda_C(M)$ if M is an integral homology sphere. In [24], Lescop defined an invariant $\lambda_{CWL}(M)$ for any closed oriented three dimensional manifold, satisfying $\lambda_{CWL}(M) = \frac{|H_1(M,\mathbb{Z})|}{2} \lambda_{CW}(M)$ whenever M is a rational homology sphere. We will follow the notation of Lescop, that is, $\lambda = \lambda_{CWL}$.

2.6.7 Proposition (Nicolaescu [55]). *Let M be a rational homology sphere with a* spin^c *structure σ and set $\mathbf{SW}^0_{M,\sigma} = \sum_{h\in H} \mathbf{sw}^0(M, h\sigma)h \in \mathbb{Q}[H]$. Then $\mathbf{SW}^0_{M,\sigma} = \mathcal{T}^0_{M,\sigma}$.*

2.6.8 Remark. Since we will only deal with rational homology spheres, we do not state the corresponding statements in [55] about three dimensional manifolds with nontrivial rational first homology.

We will now describe the identification of the normalized Seiberg–Witten invariants which we will use to prove the main theorem in section 7.

2.6.9 Proposition (Némethi [38, 32]). *Assume the notation in subsection 2.2 and subsection 2.3 and that M is a rational homology sphere. Take any $l' \in L'$ satisfying $(l', E_v) \le (Z_K, E_v)$ for all $v \in \mathcal{V}$. Then*

$$\sum_{L\ni l\not\geq 0} z_{l'+l} = \mathbf{sw}^0_M([l']\,\sigma_{\mathrm{can}}) - \frac{(-Z_K+2l')^2+|\mathcal{V}|}{8}. \tag{2.3}$$

2.6.10 Remark. In [19], László develops a general theory of multivariable power series and defines a periodic constant. In his language, eq. 2.3 means that for $h \in H$, the periodic constant of $Z_h(t)$ is the number

$$\mathbf{sw}^0_M(h\,\sigma_{\mathrm{can}}) - \frac{(-Z_K+2r_h)^2+|\mathcal{V}|}{8},$$

where r_h is the unique element in L' with $[r_h] = h$ and $0 \le m_v(r_h) < 1$ for all $v \in \mathcal{V}$.

2.7 The Seiberg–Witten invariant conjecture

In this subsection we give a very brief account of the Seiberg–Witten invariant conjecture of Némethi and Nicolaescu.

2.7.1. In [40], Némethi and Nicolaescu conjectured a topological upper bound on the geometric genus of a normal surface singularity, whose link is a rational homology sphere in terms of the normalized Seiberg–Witten invariant of the link, and the resolution graph. More precisely, the *Seiberg–Witten invariant conjecture (SWIC)* says that

$$\mathbf{sw}^0_M(\sigma_{\mathrm{can}}) - \frac{Z_K^2+|\mathcal{V}|}{8} \ge p_g, \tag{2.4}$$

with equality if the singularity is $\mathbb{Q}$-Gorenstein (in particular, Gorenstein). If the singularity is a complete intersection and the link is an integral homology sphere, then the conjecture is equivalent with the *Casson invariant conjecture (CIC)* of Neumann and Wahl [50]. Although counterexamples have been found to the SWIC (see below), it is still an interesting question to ask, under which conditions does the SWIC hold? Furthermore, although no counterexamples have been found to the CIC, it is a difficult and interesting problem to construct an isolated complete intersection singularity, whose link is an integral homology sphere, and for which the CIC has not already been determined.

2.7.2 Example. (i) Neumann and Wahl proved the CIC for weighted homogeneous singularities, suspensions of plane curves and certain complete intersections in $\mathbb{C}^4$ [50]. They also note that in the case of Brieskorn singularities, that is, hypersurface singularities given by an equation the form $x^p + y^q + z^r = 0$, the conjecture follows from work of Fintushel and Stern [10].

(ii) Némethi and Nicolaescu proved the SWIC for certain rational and minimally elliptic singularities [40], for singularities with a good $\mathbb{C}^*$ action [41] and for suspensions of irreducible plane curve singularities [42].

(iii) Némethi and Okuma proved the CIC for singularities of splice type [44], as well as the SWIC for splice quotients [43] (see [52, 51] for definitions).

(iv) Using superisolated singularities, Luengo-Velasco, Melle-Hernández and Némethi constructed counterexamples to the SWIC [27]. More precisely, they constructed hypersurface singularities (in particular, Gorenstein) for which eq. 2.4 does not hold.

(v) We prove the SWIC in the case of a hypersurface singularity with Newton nondegenerate principal part, see corollary 7.0.3.

2.8 Computation sequences

In this section we will discuss computation sequences and an upper bound on the geometric genus obtained by such sequences.

In [21], Laufer gave an algorithm to determine the minimal cycle and gave a criterion for rationality, i.e. $p_g = 0$. In [22] he used the same algorithm to find a topological characterisation of minimally elliptic singularities. This idea was generalized by Yau in [71] and by Némethi in [34] for more general elliptic singularities.

One of the main results in [46] is the existence of a computation sequence to the anticanonical cycle obtained directly from the resolution graph yielding equality in eq. 2.5 in the case of Newton nondegenerate singularities, thus giving a topological identification of the geometric genus. This result, as well as some improvements, is described in section 5.

2.8.1 Definition. Assume given a resolution graph G for a singularity $(X, 0)$. Let $Z \in L$ be an effective cycle. A *computation sequence* for Z is a sequence $Z_0, \ldots, Z_k$ so that $Z_0 = 0$, $Z_k = Z$ and for each i we have a $v(i) \in \mathcal{V}$ so that $Z_{i+1} = Z_i + E_{v(i)}$. Given such a computation sequence, its *continuation to infinity* is the sequence $(Z_i)_{i=0}^{\infty}$ recursively defined by $Z_{i+1} = Z_i + E_{v(i)}$ where we extend v to $\mathbb{N}$ by $v(i') = v(i)$ if $i' \equiv i \,(\mathrm{mod}\, k)$.

2.8.2 Theorem. *Let $Z \in L$ be an effective divisor and $(Z_i)_{i=0}^{k}$ a computation sequence for Z. Then*

$$h_Z \leq \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_{v(i)}) + 1\} \tag{2.5}$$

and we have an equality if and only if the natural maps $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i)) \to H^0(E_{v(i)}, \mathcal{O}_{E_{v(i)}}(-Z_i))$ are surjective for all i.

2.8.3 Remark. If G is numerically Gorenstein, then we have $h_{Z_K} = p_g$ by proposition 2.5.8. Therefore, eq. 2.5 gives a topological bound on the geometric genus in this case.

Proof of theorem 2.8.2. For any i, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-Z_{i+1}) \longrightarrow \mathcal{O}_{\tilde{X}}(-Z_i) \longrightarrow \mathcal{O}_{E_{v(i)}}(-Z_i) \longrightarrow 0$$

which yields the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1})) &\longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i)) \longrightarrow H^0(\tilde{X}, \mathcal{O}_{E_{v(i)}}(-Z_i)) \longrightarrow \\ \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1})) &\longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i)) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{E_{v(i)}}(-Z_i)) \longrightarrow 0. \end{aligned}$$

Denote by β_i the connection homomorphism of this sequence. We get

$$\begin{aligned} h_Z &= \dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z))} \\ &= \sum_{i=0}^{k-1} \dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))} \\ &= \sum_{i=0}^{k-1} \dim_{\mathbb{C}} H^0(\tilde{X}, \mathcal{O}_{E_{v(i)}}(-Z_i)) - \operatorname{rk} \beta_i. \end{aligned}$$

The statement now follows, since, on one hand, $E_{v(i)} \cong \mathbb{CP}^1$ and the first Chern class of $\mathcal{O}_{E_{v(i)}}(-Z_i)$ is $(-Z_i, E_{v(i)})$, and on the other, the surjectivity condition is equivalent to $\operatorname{rk} \beta_i = 0$. $\square$

2.8.4 Remark. Assume that for some i we have $(Z_i, E_{v(i)}) > 0$. Then the group $H^0(E_{v(i)}, \mathcal{O}_{E_{v(i)}}(-Z_i))$ vanishes and the surjectivity condition in theorem 2.8.2 holds automatically. Furthermore, the i^{th} summand in eq. 2.5 vanishes. Assume given a subsequence $i_1, \ldots, i_s$ of $0, \ldots, k-1$ so that if $0 \le i \le k-1$ and $i \ne i_r$ for all r, then $(Z_i, E_{v(i)}) > 0$. Then theorem 2.8.2 can be phrased entirely in terms of this subsequence, that is, the sum on the right hand side of eq. 2.5 can be taken over the subsequence (i_r) only, and the surjectivity condition is only needed for the ${i_r}^{\text{th}}$ terms.

2.9 Lattice cohomology and path lattice cohomology

In [36], Némethi introduced *lattice cohomology* as well as the related *path lattice cohomology*. In this subsection we will review how this theory relates with our results and the tools introduced so far. In general, lattice cohomology is associated with any spin$^{\mathrm{c}}$ structure on the link. For simplicity, we will assume that G is the resolution graph of a numerically Gorenstein singularity $(X, 0)$, and we will only consider invariants associated with the canonical spin$^{\mathrm{c}}$ structure σ_{can}.

2.9.1. Let $L = \mathbb{Z}\langle E_v | v \in \mathcal{V}\rangle$ be the lattice associated with a resolution graph G as in subsection 2.2. We give $L_{\mathbb{R}} = L \otimes \mathbb{R}$ the structure of a CW complex by taking as cells the cubes $\square_{l,I} = \left\{ l + \sum_{v \in I} t_v E_v \,\middle|\, \forall v \in I : \, 0 < t_v < 1 \right\}$, where $l \in L$ and $I \subset \mathcal{V}$. Let $\mathcal{Q}$ be the set of these cubes. For $l \in L$ we set $\chi(l) = (-l, l - Z_K)/2$. Note that if l is an effective cycle, then $\chi(l)$ is the Euler characteristic of the structure sheaf of the scheme defined by the ideal sheaf $\mathcal{O}_{\tilde{X}}(-l)$. The *weight function* w is defined on $\mathcal{Q}$ by setting $w(\square_{l,I}) = \max\left\{\chi(l+\sum_{v\in I'} E_v) \,\middle|\, I' \subset I\right\}$. This way, the set $S_n = \cup\{\square_{l,I} \,|\, w(\square_{l,I}) \le n\}$ is a subcomplex of $L_{\mathbb{R}}$. Note that, by negative definiteness, χ is bounded from below on L and the subcomplexes S_n are finite.

2.9.2 Definition. Let $A \subset L_{\mathbb{R}}$ be a subcomplex. The q^{th} *lattice cohomology* of the pair (A, w) is defined as

$$\mathbb{H}^q(A, w) = \bigoplus_{n\in\mathbb{Z}} H^q(A \cap S_n, \mathbb{Z}),$$

for $q \ge 0$ We also set $\mathbb{H}^*(A, w) = \oplus_{q\ge 0}\mathbb{H}^q(A, w)$. For any q and n, the inclusion $A \cap S_n \subset A \cap S_{n+1}$ induces a map on cohomology which we denote by U. This gives $\mathbb{H}^*(A, w)$ the structure of a $\mathbb{Z}[U]$ module. Similarly, we get *reduced lattice cohomology* $\mathbb{H}^*_{\text{red}}(A, w)$ by replacing cohomology H^* by reduced cohomology $\tilde{H}^*$.

For $l_1 \le l_2$ define the rectangle $R(l_1, l_2) = \cup\left\{\square_{l,I} \,\middle|\, l_1 \le l \le l + \sum_{v\in I} E_v \le l_2\right\}$.

2.9.3 Proposition (Némethi [36]). *The inclusion $R(0, Z_K) \cap S_n \subset S_n$ is a homotopy equivalence for all n. Furthermore, the complex S_n is contractible if $n > 0$.*

2.9.4 Corollary. *The group $\mathbb{H}^*_{\text{red}}(L_{\mathbb{R}}, w)$ is finitely generated.*

2.9.5 Definition. Set $m = \min \chi$ and assume that $\mathbb{H}^*(A, w)$ has finite rank. The *normalized Euler characteristic* of lattice cohomology is defined as

$$\operatorname{eu}(\mathbb{H}^*(A, w)) = -m + \sum_{q=0}^{\infty} (-1)^q \operatorname{rk} \mathbb{H}^q_{\text{red}}$$
$$\operatorname{eu}(\mathbb{H}^0(A, w)) = -m + \operatorname{rk} \mathbb{H}^0_{\text{red}}$$

2.9.6 Proposition (Némethi [38]). *We have*

$$\operatorname{eu}(\mathbb{H}^*(L_{\mathbb{R}}, w)) = \mathbf{sw}^0_M(\sigma_{\text{can}}) - \frac{Z_K^2 + |\mathcal{V}|}{8}.$$

□

2.9.7 Proposition (Némethi [36]). *Let $(Z_i)_{i=0}^k$ be a computation sequence for $Z_K \in L$ and let γ be the subcomplex of $L_{\mathbb{R}}$ consisting of the cubes $\square_{Z_i,\emptyset}$ for $0 \le i \le k$ and $\square_{Z_i,E_{v(i)}}$ for $0 \le i \le k-1$. Then*

$$\operatorname{eu}(\mathbb{H}^*(\gamma, w) = \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_{v(i)}) + 1\}.$$

Combining this result with theorem 2.8.2 and remark 2.8.3 we have the following

2.9.8 Corollary. *We have $p_g \le \min_\gamma \operatorname{eu}(\mathbb{H}^*(\gamma, w)$, where γ runs through complexes associated to any computation sequence to Z_K as in the proposition above.* □

2.10 On power series in one variable

In this subsection we recall some facts about power series in one variable and define the polynomial part of the power series expansion of a rational function. Here, as well as in the sequel, we will identify a rational function with its Taylor expansion at the origin. In particular, we will identify the localization $\mathbb{C}[t]_{(t)}$ with a subring of the ring of power series $\mathbb{C}[[t]]$. Furthermore, we will generalize these definitions to rational Puiseux series and prove a formula for these invariants for special series constructed from simplicial cones.

Némethi and Okuma introduced the periodic constant of a rational function [44, 57]. Braun and Némethi introduced the polynomial part of a rational function [4]. See also [19] for a discussion and generalization of these invariants.

Recall that a *quasipolynomial* is a function $\mathbb{Z} \to \mathbb{C}$ of the form $t \to \sum_{i=0}^{d} c_i(t)t^i$, where $c_i : \mathbb{Z} \to \mathbb{C}$ are periodic functions.

2.10.1 Proposition. *Let $P \in \mathbb{C}[t]_{(t)}$ be a rational function, regular at the origin, and consider its expansion at the origin $P(t) = \sum_{i=0}^{\infty} a_i t^i$ with $a_i \in \mathbb{C}$. Then there exists a quasipolynomial function $i \mapsto a'_i$ so that for i large enough we have $a_i = a'_i$.* □

2.10.2 Definition. Let P, a_i and a'_i be as in the proposition above. The *negative part* of P is $P^{\mathrm{neg}}(t) = \sum_{i=0}^{\infty} a'_i t^i$. The *polynomial part* of P is $P^{\mathrm{pol}}(t) = P(t) - P^{\mathrm{neg}}(t)$. The *periodic constant* of $P(t)$ is the number $\mathrm{pc}\, P(t) = P^{\mathrm{pol}}(1)$.

2.10.3 Lemma. *The polynomial part is additive. More precisely, if $P, Q \in \mathbb{C}[t]_{(t)}$, then $(P+Q)^{\mathrm{pol}} = P^{\mathrm{pol}} + Q^{\mathrm{pol}}$.*

Proof. It is clear from definition that $(P+Q)^{\mathrm{neg}} = P^{\mathrm{neg}} + Q^{\mathrm{neg}}$. The lemma follows. □

2.10.4 Remark. (i) We may write $P(t) = p(t)/q(t)$ with $p(t), q(t) \in \mathbb{C}[t]$ and $\gcd(p(t), q(t)) = 1$. Using the Euclidean algorithm, we can write $p(t) = h(t)q(t) + r(t)$ with $h(t), r(t) \in \mathbb{C}[t]$ and $\deg r(t) < \deg q(t)$ and furthermore, this presentation is unique. It is a simple exercise to show that $P^{\mathrm{neg}}(t) = r(t)/q(t)$ and $P^{\mathrm{pol}}(t) = h(t)$. In fact, $P(t) = P^{\mathrm{neg}}(t) + P^{\mathrm{pol}}(t)$ is the unique presentation of $P(t)$ as a sum of a polynomial and a fraction of negative degree.

(ii) One finds easily that $\mathbb{C}[t]_{(t)} = \mathbb{C}[t] \oplus N$ where $N = \left\{ p \in \mathbb{C}[t]_{(t)} \,\middle|\, \lim_{t\to\infty} p(t) = 0 \right\}$, and that the polynomial and negative parts are the projections to these summands. The additivity property lemma 2.10.3 follows immediately from this observation.

2.10.5. Denote by $\mathbb{C}[[t^{1/\infty}]] = \cup_{n\in\mathbb{Z}_{>0}} \mathbb{C}[[t^{1/n}]]$ the *ring of Puiseux series*. Thus, for any Puiseux series $P(t)$, there is an $N > 0$ so that $P'(t) = P(t^N) \in \mathbb{C}[[t]]$. We will say that $P(t)$ is *rational* if $P'(t)$ is rational for such a choice of N. The statements and definition above apply to this situation without much alteration. In particular, if $P(t) \in \mathbb{C}[[t^{1/\infty}]]$ is rational, and N is as above, then we set $P^{\mathrm{pol}}(t) = P'^{\mathrm{pol}}(t^{1/N})$ and $P^{\mathrm{neg}}(t) = P'^{\mathrm{neg}}(t^{1/N})$. Since $P'^{\mathrm{pol}}(t)$ is a polynomial, we find that $P^{\mathrm{pol}}(t)$ is a finite expression, that is, $P^{\mathrm{pol}}(t)$ is a *Puiseux polynomial*, $P^{\mathrm{pol}}(t) \in \mathbb{C}[t^{1/\infty}] = \cup_{n\in\mathbb{Z}_{>0}} \mathbb{C}[t^{1/n}]$.

Similarly as above, we have the field of *Laurent–Puiseux series* $\mathbb{C}((t^{1/\infty})) = \cup_{n\in\mathbb{Z}_{>0}} \mathbb{C}((t^{1/n}))$ and ring of *Laurent–Puiseux polynomials* $\mathbb{C}[t^{\pm 1/\infty}] = \cup_{n\in\mathbb{Z}_{>0}} \mathbb{C}[t^{\pm 1/n}]$.

2.10.6 Lemma. *Let $p_1, \dots, p_k \in \mathbb{Z}^n$ be linearly independent vectors, $k \geq 1$, and $\ell : \mathbb{Z}^n \to \mathbb{Q}$ a linear function taking positive values on $p_1, \dots, p_k$. Let $C = \mathbb{R}_{\geq 0}\langle p_1, \dots, p_k\rangle$ and, similarly, $C^\circ = \mathbb{R}_{>0}\langle p_1, \dots, p_k\rangle$ and define $P(t) = \sum_{p \in C \cap \mathbb{Z}^n} t^{\ell(p)} \in \mathbb{C}[[t^{1/\infty}]]$. Then, $P(t)$ is rational and $P^{\mathrm{pol}}(t) = 0$. Furthermore, if $\ell(p_i) = 1$ for all $i = 1, \dots, k$, then $((1-t)P(t))^{\mathrm{pol}} = (-1)^{k-1} \sum_{p \in S \cap \mathbb{Z}^n} t^{1-\ell(p)}$ where $S = \{p \in C^\circ \cap \mathbb{Z}^3 \,|\, \ell(p) \leq 1\}$.*

Proof. The first statement depends only on C, so by replacing the vectors p_i by suitable multiples, we may assume that $\ell(p_i) = r$ for all i for some $r \in \mathbb{Q}$. Define the half open parallelpiped $T = \sum_{i=1}^k [0, 1[p_i \subset \mathbb{R}^n$. We obtain a Puiseux polynomial $Q = \sum_{p \in T \cap \mathbb{Z}^3} t^{\ell(p)}$ of degree $< kr$ (that is, any monomial with nonzero coefficient has exponent $< kr$). Furthermore, we see that $P(t) = Q(t)(1-t^r)^k$. This shows that $P(t) \in \mathbb{C}[t]_{(t)}$ and $P^{\mathrm{neg}}(t) = P(t)$, hence the first statement.

For the second statement, we start by proving that

$$\left(\frac{t^r}{(1-t)^{k-1}}\right)^{\mathrm{pol}} = \begin{cases} 0 & \text{if } 0 \leq r < k-1, \\ (-1)^{k-1} t^{r-k+1} & \text{if } k-1 \leq r < k. \end{cases} \tag{2.6}$$

The case $0 \leq r < k-1$ is clear, since in that case, the numerator on the left has a smaller degree than the denominator. For the second case, we use induction on k. If $k = 1$, then $t^r/(1-t)^{k-1} = t^r$, so the statement is clear. Assuming $k > 1$, we find

$$\left(\frac{t^r}{(1-t)^{k-1}}\right)^{\mathrm{pol}} = \left(-\frac{t^{r-1} - t^r}{(1-t)^{k-1}}\right)^{\mathrm{pol}} = \left(-\frac{t^{r-1}}{(1-t)^{k-2}}\right)^{\mathrm{pol}} = (-1)^{k-1} t^{r-k+1}$$

using the additivity of the polynomial part and the induction hypothesis. The result now follows, since

$$(1-t)P(t) = \sum_{p \in T \cap \mathbb{Z}^n} \frac{t^{\ell(p)}}{(1-t)^{k-1}}$$

where T is as before, because the set S is in bijection with $\{p \in T \,|\, \ell \geq k-1\}$ via $p \mapsto \sum_i p_i - p$, and we have $\ell(p) - k + 1 = 1 - \ell(\sum_i p_i - p)$. $\square$

2.10.7 Example. (i) Let $A = \oplus_{i=0}^\infty A_k$ be the coordinate ring of an affine variety $X \subset \mathbb{C}^N$ with a good $\mathbb{C}^*$ action as in [59] (with the origin a fixed point) and $P(t) = \sum_{i=0}^\infty \dim_{\mathbb{C}} A_i t^i$ the associated Poincaré series. In [59], Pinkham shows that $P(t)$ can be described in therms of the link at the origin and that $p_g = \mathrm{pc}\, P(t)$.

(ii) Let G be a negative definite graph as in subsection 2.2 and assume that G satisfies the *semigroup condition* and the *congruence condition* described in [51] (or, equivalently, the end curve condition, see [53, 58]). Neumann and Wahl [51] constructed a singularity (more precisely, a set of singularities forming an equisingular deformation) whose topological type is given by G. Such a singularity is called a *splice quotient singularity*. If $v \in \mathcal{V}$ and G_i are the components of the complement of v in G, then these subgraphs satisfy the same conditions. Okuma showed [57] that the geometric genus of a splice quotient singularity is the sum of the geometric genera of splice quotient singularities with graphs G_i plus an error term, which is the periodic constant associated with a series in one variable obtained from G and v. More precisely, this series has two descriptions. On one hand it is the Poincaré series associated with the graded ring associated with the divisorial filtration on the local ring of the singularity given by the divisor E_v. On the other hand, it is the function $Z_0^v(t_v)$, obtained from the topological zeta function $Z_0(t)$ (see subsection 2.3) by the restriction $t_w = 1$ for $w \neq v$.

(iii) In [4], Braun and Némethi obtain a surgery formula for the normalized Seiberg–Witten invariant associated with the graphs G, G_i in the previous example, but with no assumption on the graph other than negative definiteness. In place of the geometric genus, the formula contains a normalized version of the Seiberg–Witten invariant of the canonical spin^c structure on the associated three dimensional manifold. The error term is, as in the previous example, the periodic constant of $Z_0^v(t_v)$.

2.11 The spectrum

In this section we will recall some facts about the spectrum, a numerical invariant coming from Hodge theory. Its construction would require a lengthier treatment than is possible here, so we only mention the main results required. The most important fact we need about the spectrum is proposition 2.11.9, which allows us to calculate part of the spectrum from the Newton diagram. In section 6, we will show how to recover this part of the spectrum directly, given only the knowledge of the resolution graph, as well as the divisor of the function $x_1x_2x_3$.

2.11.1. We start with a very small account of the results leading to the mixed Hodge structure on the cohomology groups of the Milnor fiber. Mixed Hodge structures were introduced by Deligne in [5, 6] where he constructs a mixed Hodge structure on the cohomology groups of arbitrary algebraic varieties, generalizing the Hodge decomposition on Kähler manifolds [16]. Previously, Griffiths, Schmid [12, 13, 14, 65] and others had studied variations of Hodge structures arising from deformations of complex manifolds, as well as the case of flat maps, possibly with singular fibers. For these, a limit of the Hodge structures appears (in a suitable sense), but this must be viewed as a mixed Hodge structure, rather than a pure Hodge structure. In [66], Steenbrink considers the same problem from a different viewpoint and constructs a mixed Hodge complex calculating this limit. In [67], these results are combined with others to construct a limit mixed Hodge structure on the cohomology groups of the Milnor fiber of an isolated hypersurface singularity.

2.11.2. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a singular map germ defining an isolated hypersurface singularity $(X, 0)$ in $\mathbb{C}^3$. Assume that $Y \subset \mathbb{C}^3$ is a subset yielding a *good representative* of the Milnor fibration, where $D \subset \mathbb{C}$ is some small disc (see e.g. [26]). Setting $D^* = D \setminus \{0\}$ and $Y^* = Y \setminus f^{-1}(0)$ we obtain a locally trivial fiber bundle $Y^* \to D^*$ whose fiber is the Milnor fiber. For a $t \in D^*$, denote by $m_t : Y_t \to Y_t$ the geometric monodromy, and $T_t : H^2(Y_t, \mathbb{C}) \to H^2(Y_t, \mathbb{C})$ the induced map on cohomology, the algebraic monodromy. We can assume that together, these form a diffeomorphism $m^* : Y^* \to Y^*$.

Take $\tilde{D}^*$ as the universal covering space of the punctured disc, and set $Y_\infty = Y^* \times_{D^*} \tilde{D}^*$. Concretely, we may take $\tilde{D}^*$ as an upper half plane, with the covering map given by the exponential function. We obtain canonical maps $k : Y_\infty \to Y$ and $f_\infty : Y_\infty \to D^*$, as well as monodromy transformations $m_\infty : Y_\infty \to Y_\infty$ and $\tilde{m}^* : \tilde{D}^* \to \tilde{D}^*$ satisfying $f_\infty m_\infty = \tilde{m}^* f_\infty$ and $k m_\infty = m^* k$. This is summarized in the following diagram.

$$
\begin{tikzcd}
 & Y^* \arrow[loop left, "m^*"] \arrow[d, hook] & \\
Y_\infty \arrow[loop left, "m_\infty"] \arrow[r, "k"] \arrow[d, "f_\infty"'] & Y \arrow[d, "f"'] & X \arrow[l, hook', "i"'] \arrow[d] \\
\tilde{D}^* \arrow[loop left, "\tilde{m}^*"] \arrow[r] & D & \{0\} \arrow[l, hook']
\end{tikzcd}
\tag{2.7}
$$

The space $\tilde{D}^*$ is a half plane in $\mathbb{C}$, and so, in particular, it is contractible. Therefore, the space $Y_\infty \cong Y_t \times \tilde{D}^*$ has the same homotopy type as any Milnor fiber Y_t. Furthermore, this homotopy equivalence is determined uniquely modulo the monodromy.

2.11.3 Proposition (Monodromy theorem [18, 65])**.** *The eigenvalues of the monodromy operator $T_\infty : H^2(Y_\infty, \mathbb{C}) \to H^2(Y_\infty, \mathbb{C})$ are roots of unity, that is, there is an $N > 0$ so that T_∞^N is unipotent. Furthermore, for such an N, we have $(T_\infty^N - \mathrm{id})^3 = 0$.*

2.11.4. In [67], Steenbrink constructs a mixed Hodge structure on the *vanishing cohomology* $H^2(Y_\infty)$. This means that on $H^2(Y_\infty, \mathbb{Q})$ one has the *weight filtration*

$$0 = W_0 H^2(Y_\infty, \mathbb{Q}) \subset \ldots \subset W_{2n} H^2(Y_\infty, \mathbb{Q}) = H^2(Y_\infty, \mathbb{Q})$$

and on $H^2(Y_\infty, \mathbb{C})$, the *Hodge filtration*

$$H^2(Y_\infty, \mathbb{C}) = F_0 H^2(Y_\infty, \mathbb{C}) \supset \ldots \supset F_n H^2(Y_\infty, \mathbb{C}) = 0$$

where in our case, $n = 2$. Furthermore, these susbspaces are invariant under the semisimple part of the monodromy operator T_∞. The filtrations induce graded objects

$$\mathrm{Gr}_k^W H^2(Y_\infty, \mathbb{Q}) = \frac{W_k H^2(Y_\infty, \mathbb{Q})}{W_{k-1} H^2(Y_\infty, \mathbb{Q})}, \quad \mathrm{Gr}_F^p H^2(Y_\infty, \mathbb{C}) = \frac{F^p H^2(Y_\infty, \mathbb{C})}{F^{p+1} H^2(Y_\infty, \mathbb{C})}.$$

Furthermore, as a subquotient of $H^2(Y_\infty, \mathbb{C})$, the space $\mathrm{Gr}_k^W H^2(Y_\infty, \mathbb{Q}) \otimes \mathbb{C}$ inherits the Hodge filtration making it a Hodge structure of weight k. For each of these spaces, we denote by $(\cdot)_\lambda$ the generalized eigenspace of T_∞ with eigenvalue λ.

2.11.5 Definition. The *spectrum* of an isolated hypersurface singularity defined by $f \in \mathcal{O}_{\mathbb{C}^3,0}$ is the element

$$\mathrm{Sp}(f, 0) = \sum_\lambda \sum_p \dim_{\mathbb{C}} (\mathrm{Gr}_F^p H^2(Y_\infty, \mathbb{C}))_\lambda \left(\frac{\log \lambda}{2\pi i} + n - p \right) \in \mathbb{Z}[\mathbb{Q}]. \tag{2.8}$$

where we choose $-1 < \frac{\log \lambda}{2\pi i} \leq 0$, and (a) denotes the element corresponding to $a \in \mathbb{Q}$ in the group ring $\mathbb{Z}[\mathbb{Q}]$. This choice is made possible precisely by proposition 2.11.3 For any subset $I \subset \mathbb{Q}$ we define $\mathrm{Sp}_I(f,0) = \pi_I(\mathrm{Sp}(f,0))$, where $\pi_I : \mathbb{Z}[\mathbb{Q}] \to \mathbb{Z}[\mathbb{Q}]$ is the projection sending (a) to (a) if $a \in I$, but to 0 if $a \notin I$. For simplicity, we also set $\mathrm{Sp}_{\leq 0}(f,0) = \mathrm{Sp}_{]-\infty,0]}(f,0)$.

Since the coefficients in eq. 2.8 are nonnegative integers, we may also write $\mathrm{Sp}(f,0) = \sum_{j=1}^{\mu}(l_j)$ where $l_1, \dots, l_\mu \in \mathbb{Q}$ satisfy $l_1 \leq \dots \leq l_\mu$. Here, $\mu = \dim_{\mathbb{C}}(Y_\infty, \mathbb{C})$ is the Milnor number.

2.11.6. The mixed Hodge structure on the vanishing cohomology induces *Hodge numbers*

$$h^{p,q} = \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^2(Y_\infty, \mathbb{C}). \tag{2.9}$$

The weight and Hodge filtrations are invariant under the semisimple part of the monodromy, see [67] remark (3.11). In particular, it induces an action on the space on the right hand side of eq. 2.9. This gives rise to *equivariant Hodge numbers*

$$h_\lambda^{p,q} = \dim_{\mathbb{C}} \left(\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^2(Y_\infty, \mathbb{C})\right)_\lambda \tag{2.10}$$

for $\lambda \in \mathbb{C}$. Equivalently to the definition above, we now have

$$\mathrm{Sp}(f,0) = \sum_{p,q,\lambda} h_\lambda^{p,q} \left(\frac{\log \lambda}{2\pi i} + n - p\right) = \sum_{\alpha \in \mathbb{Q}} \sum_{q \in \mathbb{Z}} h_{\exp(2\pi i \alpha)}^{n+\lfloor -\alpha \rfloor, q}(\alpha).$$

2.11.7 Remark. The spectrum is an invariant that depends only on the Hodge filtration. A stronger invariant, the *spectral pairs*, take the weight filtration into account as well. In fact, the spectral pairs encode the same data as the equivariant Hodge numbers. We will, however, not make any use of the spectral pairs.

2.11.8 Proposition ([64] (7.3)). *We have the following properties.*

(i) The spectrum is symmetric around $\frac{1}{2}$. More precisely, we have $\mathrm{Sp}(f,0) = \iota\,\mathrm{Sp}(f,0)$, where $\iota : \mathbb{Z}[\mathbb{Q}] \to \mathbb{Z}[\mathbb{Q}]$ is the group automorphism sending (a) to $(\frac{1}{2} - a)$ for $a \in \mathbb{Q}$.

(ii) The spectrum is contained in the interval $]-1,2[$. More precisely, for every monomial (a) in the sum eq. 2.8 with nonzero coefficient we have $-1 < a < 2$.

□

2.11.9 Proposition (Saito [63]). *Let $f \in \mathcal{O}_{\mathbb{C}^3,0}$ define an isolated hypersurface singularity $(X,0) \subset (\mathbb{C}^3,0)$. Assume further more that f has Newton nondegenerate principal part (for definitions of diagrams and nondegeneracy, see subsection 3.1, for the Newton filtration, see subsection 3.5). The part of the spectrum lying in $]-1,0]$ is given by the Newton weight function of monomials containing all three variables which are under the Newton diagram. That is, we have $\mathrm{Sp}_{\leq 0}(f,0) = \sum_{p \in \mathbb{Z}^3_{>0} \cap \Gamma_-(f)}(\varphi(x^p) - 1)$.*

2.12 Statement of results

Assume that $f \in \mathcal{O}_{\mathbb{C}^3,0} = \mathbb{C}\{x_1, x_2, x_3\}$ is the germ of a holomorphic function in three variables defining an isolated hypersurface singularity $(X,0)$ at the origin. Assume, furthermore, that f has Newton nondegenerate principal part (see subsection 3.1) and that that the link M of $(X,0)$ is a rational homology sphere. We denote by p_g the geometric genus of $(X,0)$.

2.12.1 Theorem. I *There exists as computation sequence* $(Z_i^I)_{i=0}^k$ *to* Z_K *on the minimal good resolution graph of* M *satisfying*

$$p_g = \sum_{i=0}^{k} \max\{0, (-Z_i^I, E_{v(i)}) + 1\} = \mathbf{sw}_M^0(\sigma_{\mathrm{can}}) - \frac{Z_K + |\mathcal{V}|}{8}.$$

Furthermore, this computation sequence can easily be computed using the minimal resolution graph, see definition 5.2.2.

II *Assuming that the Newton diagram* $\Gamma(f)$ *is convenient (see definition 3.1.3) and that* G *is the resolution graph obtained from Oka's algorithm using this diagram (see 3.2.3), there exists a computation sequence* $(Z_i^{II})_{i=0}^k$ *to* $\mathrm{wt}(f)$ *satisfying*

$$P_X^{\mathcal{A}}(t) = \sum_{i=0}^{\infty} \max\{0, (-Z_i^{II}, E_{v(i)}) + 1\} t^{r_i}$$

where $P_X^{\mathcal{A}}(t)$ *is the Poincaré series associated to the Newton filtration on* $\mathcal{O}_{X,0}$ *(see subsection 3.5),* $(Z_{i=0}^{II})^{\infty}$ *is the continuation of* (Z^{II}) *to infinity as in definition 2.8.1 and we set* $r_i = m_{v(i)}(Z_i^{II})$ *for each* $i \geq 0$. *Furthermore, the part of the spectrum* $\mathrm{Sp}_{\leq 0}(f, 0)$ *is obtained from this series by the equality*

$$\mathrm{Sp}_{\leq 0}(f, 0) = P_X^{\mathcal{A},\mathrm{pol}}(t^{-1}),$$

where we identify the ring of Laurent–Puiseux series $\mathbb{Z}[t^{\pm 1/\infty}]$ *with the group ring* $\mathbb{Z}[\mathbb{Q}]$. *Furthermore, this sequence can easily be computed, assuming only the knowledge of* G*, the resolution graph, and the cycle* $\mathrm{wt}(x_1x_2x_3)$*, see definition 5.2.2.*

III *Assuming that the Newton diagram* $\Gamma(f)$ *is convenient (see definition 3.1.3) and that* G *is the resolution graph obtained from Oka's algorithm using this diagram (see 3.2.3), there exists a computation sequence* $(Z_i^{III})_{i=0}^k$ *to* $x(Z_K - E)$ *satisfying*

$$\mathrm{Sp}_{\leq 0}(f, 0) = \sum_{i=0}^{k-1} \max\{0, (Z_i^{III}, E_{v(i)}) + 1\}(r_i) \in \mathbb{Z}[\mathbb{Q}],$$

where, for each i *we set*

$$r_i = \frac{m_{v(i)}(Z_i^{III}) + \mathrm{wt}_{v(i)}(x_1x_2x_3)}{\mathrm{wt}_{v(i)}(f)}.$$

Furthermore, this sequence can easily be computed, assuming only the knowledge of G*, the resolution graph, and the cycle* $\mathrm{wt}(x_1x_2x_3)$*, see definition 5.2.2.*

Proof. See theorems 6.1.1, 6.2.1 and 7.0.2. □

2.12.2 Remark. The result $p_g = \sum_{i=0}^{k} \max\{0, (-Z_i^I, E_{v(i)}) + 1\}$ in I in the theorem above can be found in a joint article of Némethi and the author [46].

3 Newton diagrams and nondegeneracy

In this section we will recall the definition of a Newton diagram associated with a function $f \in \mathcal{O}_{\mathbb{C}^3,0}$, the nondegeneracy condition and some important properties of singularities defined by nondegenerate functions.

In what follows, f is a function germ around the origin in $\mathbb{C}^3$ and $(X,0)$ is the germ of the zero set of f. We will assume that X has an isolated singularity at the origin (see 3.1.2), that f has Newton nondegenerate principal part (definition 3.1.1) and that the link is a rational homology sphere.

3.1 Diagrams and nondegeneracy

Write $f = \sum_{p\in\mathbb{N}^3} a_p x^p$. We define the *support* of f as $\mathrm{supp}(f) = \{p \in \mathbb{N}^3 \,|\, a_p \neq 0\}$. The *Newton polytope* $\Gamma_+(f)$ of f is the convex closure of $\cup_{p\in\mathrm{supp}(f)} p + \mathbb{R}^3_{\geq 0}$. The *Newton diagram* $\Gamma(f)$ of f is the union of compact two dimensional faces of the Newton polytope. Here, a face $F \subset \Gamma_+(f)$ means the minimal set of any linear function $\mathbb{R}^3 \to \mathbb{R}$. We also denote by $\Gamma_-(f)$ the union of segments joining the origin in $\mathbb{R}^3$ with $\Gamma(f)$.

3.1.1 Definition. Let $F \subset \Gamma(f)$ be a compact face of the Newton polytope and define $f_F(x) = \sum_{p\in F} a_p x^p$. We say that f is *nondegenerate with respect to* F if the set of equations $\frac{\partial}{\partial x_i} f_F = 0$ has no solution in $(\mathbb{C}^*)^3$. We say that f has *Newton nondegenerate principal part* if f is nondegenerate with respect to every nonempty face of $\Gamma(f)$.

3.1.2 Lemma (Koushnirenko [17] 1.13)**.** *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ define a singularity $(X,0)$ and assume that f has Newton nondegenerate principal part. Then X has an isolated singularity at 0 if and only if $\Gamma(f)$ contains a point on each coordinate hyperplane and a point in distance at most 1 from each coordinate axis.* □

3.1.3 Definition. We say that $f \in \mathcal{O}_{\mathbb{C}^3,0}$ is *convenient* (f. *commode*) if any of the following equivalent conditions is fulfilled.

- ❀ $\mathrm{supp}(f)$ contains an element of each coordinate axis.
- ❀ The set $\mathbb{R}^3_{\geq 0} \setminus \Gamma_+(f)$ is bounded.
- ❀ $\mathbb{R}_{\geq 0}\Gamma(f) = \mathbb{R}^3_{\geq 0}$.

3.2 Oka's algorithm

In this section we will use the Newton diagram $\Gamma(f)$ to construct a graph G. Oka proved that this graph is the graph of a resolution of X, obtained by a toric modification of $\mathbb{C}^3$ [56]. We will use the notation from 2.2 for this resolution.

Recall that for integers $b_1, \ldots, b_s$ we have the *negative continued fraction*

$$[b_1, \ldots, b_s] = b_1 - \cfrac{1}{b_2 - \ddots}.$$

Further, the string $b_1, \ldots, b_s$ is referred to as the *negative continued fraction expansion* of the rational number above. If we require $b_j \geq 2$ for $j \geq 2$, then the expansion is unique. As we will never make use of positive continued fraction, we will often simply say continued fraction. See [60] for a detailed discussion of continued fractions and how they relate to the topology of surface singularities.

The statements in the following definition are not difficult to prove.

3.2.1 Definition. Let A be a free abelian group of finite rank and take distinct primitive elements $a, b \in A$.

- The *determinant* $\alpha(a, b)$ of a, b is the greatest common divisor of maximal minors of the matrix whose rows are given by the coordinate vectors of a and b with respect to some basis of A.
- If $\alpha(a, b) > 1$, then we define the *denominator* $\beta(a, b)$ of a, b as the unique integer $0 \leq \beta(a, b) < \alpha(a, b)$ for which $\beta(a, b)a + b$ has content $\alpha(a, b)$.
- If $\alpha(a, b) = 1$, we choose the denominator to be $\beta(a, b) = 1$ or $\beta(a, b) = 0$.
- If $\alpha(a, b) > 1$, then the *selfintersection numbers* associated with a, b are defined as $-b_1, \ldots, -b_s$, where $b_1, \ldots, b_s$ is the continued fraction expansion of $\alpha(a, b)/\beta(a, b)$.
- The *canonical primitive sequence* associated with a, b is the unique sequence $a_1, \ldots, a_s \in A$ satisfying $a_{i-1} - b_i a_i + a_{i+1} = 0$ for $i = 1, \ldots, s$, where $a_0 = a$ and $a_{s+1} = b$.
- If $\alpha(a, b) = 1$ and we choose $\beta(a, b) = 1$, then the selfintersection numbers associated with a, b consist of a single -1, and the canonical primitive sequence is $a_1 = a + b$. If we choose $\beta(a, b) = 0$, then both sequences are empty.

We refer to $\alpha(a, b)/\beta(a, b)$ as the *fraction* associated with $a, b \in A$.

3.2.2 Remark. (i) We have $\gcd(\alpha(a, b), \beta(a, b)) = 1$. Thus, the fraction associated with a, b determines the determinant and the denominator.

(ii) The canonical primitive sequence can be calculated as follows. First, we have $a_1 = (\beta(a, b)a + b)/\alpha(a, b)$. Then the other elements can be calculated recursively by the defining equations.

3.2.3. We are now ready to construct the graph G. First, let $\mathcal{N}^*$ be a set indexing the two dimensional faces of $\Gamma_+(f)$, that is, let $\{F_n \,|\, n \in \mathcal{N}^*\}$ be the set of two dimensional faces of $\Gamma_+(f)$. Let $\mathcal{N}$ be the subset of $\mathcal{N}^*$ corresponding to compact faces. For each $n \in \mathcal{N}^*$ let ℓ_n be the unique integral primitive linear function on $\mathbb{R}^3$ having F_n as its minimal set on $\Gamma_+(f)$. For any $n \in \mathcal{N}$ and $n' \in \mathcal{N}^*$, let $t_{n,n'}$ be the one dimensional combinatorial volume of $F_n \cap F_{n'}$. This is the same as the number of components of $F_n \cap F_{n'} \setminus \mathbb{Z}^3$. We also define $\alpha_{n,n'} = \alpha(\ell_n, \ell_{n'})$ and $\beta_{n,n'} = \beta(\ell_n, \ell_{n'})$, where, if $\alpha_{n,n'} = 1$, we choose $\beta_{n,n'} = 0$ if $n' \in \mathcal{N}$, but $\beta_{n,n'} = 1$ if $n' \in \mathcal{N}^* \setminus \mathcal{N}$.

The graph G^* is obtained as follows. First take $\mathcal{N}^*$ as vertex set. Then, for any $n \in \mathcal{N}$ and $n' \in \mathcal{N}^*$, add $t_{n,n'}$ copies of the bamboo depicted in fig. 1.

Let $\ell_{v_1}, \ldots, \ell_{v_s}$ be the canonical primitive sequence associated with $\ell_n, \ell_{n'}$. We then have elements ℓ_v associated with all vertices of the graph G^*. Let $\mathcal{V}^*$ be the set of vertices of G^* and for $v \in \mathcal{V}^*$, let $\mathcal{V}^*_v$ be the set of neighbours of v in G^*.

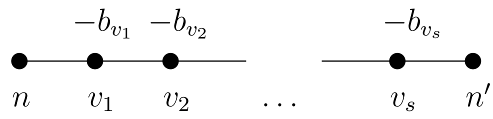

Figure 1: A bamboo.

Let $\mathcal{V}$ be the set of vertices not in $\mathcal{N}^*\setminus\mathcal{N}$. Then, define G as the subgraph of G^* generated by the vertex set $\mathcal{V}$. The vertices $v_1,\ldots,v_s$ (as in fig. 1) are labelled with the selfintersection numbers associated with $\ell_n,\ell_{n'}$, taken as (primitive) elements of $\mathrm{Hom}(\mathbb{Z}^3,\mathbb{Z})$. For $n\in\mathcal{N}$ we define the selfintersection number $-b_n$ as the unique solution of the equation

$$-b_n\ell_n+\sum_{u\in\mathcal{V}_n^*}\ell_u=0. \tag{3.1}$$

Thus, for every $v\in\mathcal{V}$ we have a selfintersection number $-b_v$. Furthermore, by the definition of b_v for $v\in\mathcal{V}\setminus\mathcal{N}$, eq. 3.1 holds with n replaced by v.

For G to be a plumbing graph, we must provide genera $[g_v]$ for all $v\in\mathcal{V}$. For $n\in\mathcal{N}$, let g_n be the number of integral points in the relative interior of the polygon F_n. All other vertices get genus 0.

3.2.4 Definition. In addition to the linear functions ℓ_v for $v\in\mathcal{V}^*$ defined above, let ℓ_c be the standard coordinate functions for $c=1,2,3$, that is, $\ell_c(p)=p_c$ for $p=(p_1,p_2,p_3)\in\mathbb{R}^3$.

3.2.5 Definition. For $n\in\mathcal{N}$, let $\mathcal{N}_n^*=\{n'\in\mathcal{N}^*\,|\,t_{n,n'}>0\}$ and $\mathcal{N}_n=\mathcal{N}_n^*\cap\mathcal{N}$. If $n\in\mathcal{N}$, $n'\in\mathcal{N}_n^*$ and $\beta_{n,n'}\neq 0$, let $u_{n,n'}=v_1$ as in fig. 1. If $\beta_{n,n'}=0$, let $u_{n,n'}=n'$.

3.2.6 Remark. (i) Note that $\beta_{n,n'}=0$ can only happen if $n'\in\mathcal{N}$, thus we always have $u_{n,n'}\in\mathcal{V}$. In particular, we have $\mathcal{V}_n^*=\mathcal{V}_n$ for $n\in\mathcal{N}$.

(ii) If $t_{n,n'}>1$, we must a neighbour $u_{n,n'}$ out of a set of $t_{n,n'}$ elements. By construction, however, the functional ℓ_u is well defined, for any such choice. The numbers $m_u(\psi(l))$ (see lemma 7.1.6) and $m_u(Z)$, where $x(Z)=Z$ (see subsection 5.1), are also well defined in this case.

3.2.7 Remark. For $n\in\mathcal{N}$, the existence of b_n is not obvious, but can be seen as follows. Let H be the hyperplane in $\mathbb{R}^3$ defined by $\ell_n=m$, where m is the value of ℓ_n on F_n. It follows from the definition of the canonical primitive sequence that for any $u\in\mathcal{V}_n$, the affine function $\ell_u|_H$ is in fact primitive, and its minimal set on F_n is $F_n\cap F_{n'}$, where u is assumed to lie on a bamboo connecting n and $n'\in\mathcal{N}^*$. We now see that there is a natural correspondence between the neighbours $u\in\mathcal{V}_n$ and the primitive segments of the boundary ∂F_n. It is simple to show that under these conditions, the sum $\sum_u\ell_u|_H$ is constant (see e.g. proof of theorem 4.2.2). Since ℓ_n is by definition also constant on H, the existence of b_n follows. Furthermore, since ℓ_n is primitive, we have $b_n\in\mathbb{Z}$. Finally, since all ℓ_v are positive on the open positive quadrant, we must have $b_n>0$. In [56], the number appears as the selfintersection of a divisor and eq. 3.1 is derived from this.

3.2.8 Lemma. *Let $n \in \mathcal{N}$ and $n' \in \mathcal{N}_n$. Then, for $u = u_{n,n'}$ we have $\alpha_{n,n'} m_u(Z_K - E) = \beta_{n,n'} m_n(Z_K - E) + m_{n'}(Z_K - E)$. Similarly, if $n' \in \mathcal{N}_n^* \setminus \mathcal{N}$, then $\alpha_{n,n'} m_u(Z_K - E) = \beta_{n,n'} m_n(Z_K - E) - 1$.*

Proof. This follows from the more general lemma 7.1.2, since $(Z_K - E, E_v) = 0$ if $\delta_v = 2$. $\square$

3.2.9 Proposition. *Take $f \in \mathcal{O}_{\mathbb{C}^3,0}$ as above with Newton nondegenerate principal part defining an isolated hypersurface singularity $(X, 0)$. The link of $(X, 0)$ is a rational homology sphere if and only if $\Gamma(f) \cap \mathbb{Z}^3_{>0} = \emptyset$.*

Proof. Let g and c be as in proposition 2.1.5. We see immediately that $g = 0$ if and only if for each $n \in \mathcal{N}$, the face F_n contains no integral points in its relative interior.

If $c \neq 0$, then we must have at least one of the following possibilities: there are $n_1, n_2 \in \mathcal{N}$ with $t_{n_1,n_2} > 1$, or, there are $n_1, \ldots, n_s \in \mathcal{N}$ so that $t_{n_i,n_{i+1}} \neq 0$ for $i = 1, \ldots, s$ (where we set $n_{s+1} = n_1$) and $\cap_{i=1}^s F_{n_i}$ is a zero dimensional face of $\Gamma(f)$. In the first case, $F_{n_1} \cap F_{n_2} \subset \Gamma(f)$ contains an integral point with positive coordinates and in the second case the point in $\cap_{i=1}^s F_{n_i}$ is such a point.

Assume now that $(X, 0)$ is isolated and has rational homology sphere link, then $t_{n,n'} \leq 1$ for all $n, n' \in \mathcal{N}$ and $g_n = 0$ for all $n \in \mathcal{N}$, so p must lie on the boundary of $\Gamma(f)$. But every segment of the boundary of $\Gamma(f)$ which does is not contained in some coordinate hyperplane has the form $[(a, 0, b), (0, 1, c)]$ for some $a, b, c \in \mathbb{N}$ modulo permutation of coordinates. But then all integral points on the boundary of $\Gamma(f)$ lie on some coordinate hyperplane, and we have $\Gamma(f) \cap \mathbb{Z}^3_{>0} = \emptyset$. $\square$

We end this subsection with the following result which can greatly simplify calculations.

3.2.10 Proposition. *Let $[p_1, p_2] \subset F_n$ be an edge of one of the faces of the Newton diagram $\Gamma(f)$, thus, $[p_1, p_2] = F_n \cap F_{n'}$ for some $n' \in \mathcal{N}_n^*$. Let $q_1, q_2 \in \partial F_n \cap \mathbb{Z}^3$ so that $[p_1, q_1]$ and $[p_2, q_2]$ are the primitive segments adjacent to $[p_1, p_2]$ in ∂F_n and set $\alpha_1 = \ell_{n'}(q_1 - p_1)$ and $\alpha_2 = \ell_{n'}(q_2 - p_2)$. If p_1 is a regular vertex of F_n, then $\alpha_{n,n'} = \alpha_1 | \alpha_2$ (see definition 4.1.2 for regular vertices).*

Proof. A simple calculation shows that $\alpha_{n,n'}$ can be identified as the content of the affine function $\ell_{n'}|_{H_n^=(\mathrm{wt}(f))}$, that is, the smallest positive integer c for which there is an integer $0 \leq r < c$ and an integral functional $\ell : H_n^=(\mathrm{wt}(f)) \to \mathbb{R}$ so that $\ell_{n'}|_{H_n^=(\mathrm{wt}(f))} = c\ell + r$. It follows that there are $a_1, a_2 \in \mathbb{N}$ so that $\alpha_1 = a_1 \alpha_{n,n'}$ and $\alpha_2 = a_2 \alpha_{n,n'}$. Since p_1 is a regular vertex of F_n, the points p_1, p_2, q_1 form an integral affine basis for $H_n^=(\mathrm{wt}(f))$, hence $a_1 = 1$, and so $\alpha_1 = \alpha_{n,n'}$ and $\alpha_2 = a_2 \alpha_{n,n'}$. $\square$

3.2.11 Remark. Assume that $(X, 0)$ is as in proposition 3.2.9 and that the link of $(X, 0)$ is a rational homology sphere. Then, by corollary 4.1.7, any edge of a face F_n of the Newton diagram contains a regular vertex of F_n as an endpoint.

3.2.12 Example. Let $f(x_1, x_2, x_3) = x_1^4 + x_1^3 x_2^2 + x_2^{10} + x_1^2 x_3^3 + x_2^3 x_3^4 + x_3^8$. A simple calculation shows that the Newton polygon is given by the inequalities

$$\langle (11,5,7), \cdot \rangle \geq 43, \quad \langle (6,3,4), \cdot \rangle \geq 24, \quad \langle (32,12,21), \cdot \rangle \geq 120, \quad \langle (6,3,4), \cdot \rangle \geq 48,$$

as a subset of the positive octant. In this case, the set $\mathcal{N}$ contains four elements, and the set $\mathcal{N}^*$ contains three elements, one corresponding to each coordinate hyperplane. On the left hand side of fig. 2, we see the Newton diagram $\Gamma(f)$. On the right hand side, circles represent compact faces of the Newton polygon and crosses represent noncompact ones. The segments joining n and n' in this picture represent $t_{n,n'}$. By calculation, we obtain the plumbing graph shown

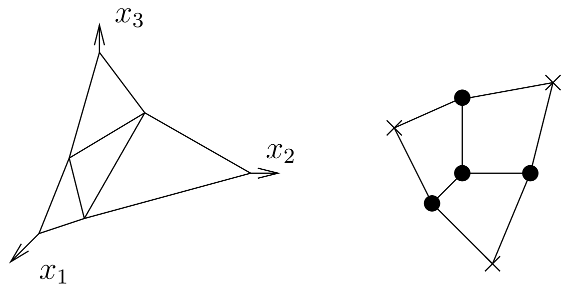

Figure 2: A Newton diagram and its dual graph in the plane.

in fig. 3, with additional vertices corresponding to the elements of $\mathcal{N}^* \setminus \mathcal{N}$.

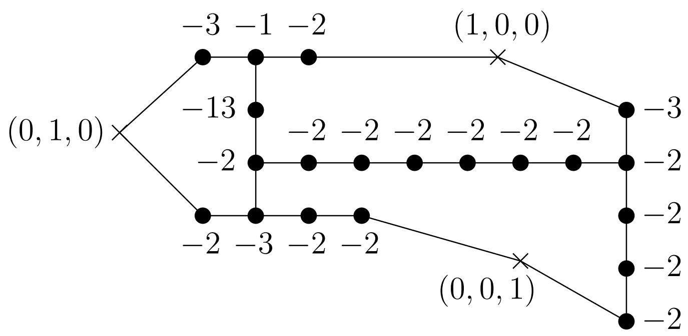

Figure 3: A plumbing graph obtained by Oka's algorithm.

3.3 On minimality

In this subsection, we will recall some results on minimality of plumbing graphs on one hand, and of Newton diagrams on the other. In [17], Kouchnirenko introduces the condition of convenience, (see definition 3.1.3), the assumption of which can be of great convenience, but does not actually reduce the generality when working with isolated singularities. This is because for a given $f \in \mathcal{O}_{\mathbb{C}^3,0}$ with Newton nondegenerate principal part, defining an isolated singularity, the function $f + \sum_{i=1}^3 x_i^d$, for d large enough, defines an analytically equivalent singularity, and the Newton diagram of the new function is convenient.

3.3.1. In [48], Neumann showed that if M is an oriented three dimensional manifold which can be represented by a plumbing graph, then there is a unique minimal graph representing M. If the intersection matrix associated with the plumbing graph G is negative definite, then minimality, in this sense, means that G contains no vertex v with $\delta_v \leq 2$ and $E_v^2 = -1$. A minimal representative can be obtained by blowing down -1 curves whenever possible.

3.3.2. In [3], Braun and Némethi provided a condition for a Newton diagram to be minimal and showed that every singularity defined by a function with a Newton nondegenerate principal part is topologically equivalent to one whose Newton diagram is minimal. In general, these minimal diagrams are not convenient. They do, however, have the advantage that if one applies Oka's algorithm on a minimal diagram as in 3.2.3, then the output is a minimal plumbing graph.

The following proposition essentially repeats some of the results of [3]:

3.3.3 Proposition. *Let $f \in \mathcal{O}_{\mathbb{C}^3,0}$ have Newton nondegenerate principal part, defining an isolated singularity at* 0 *with a rational homology sphere link, which is not an A_n singularity. Let G be the resolution graph constructed in 3.2.3 from $\Gamma(f)$.*

(i) There is a bijective correspondence between nodes $n \in \mathcal{N}$ in G and two dimensional faces $F_n \subset \Gamma(f)$ and for each $n \in \mathcal{N}$ there is a bijection between neighbours $u \in \mathcal{V}_n$ of n and primitive segments of the boundary ∂F_n of F_n. In particular, $\mathrm{Vol}_1(\partial F_n) = \delta_n$.

(ii) If $\Gamma(f)$ is minimal in the sense of definition 3.3.5 of [3], then G is minimal in the sense of 3.3.1

Proof. (i) follows directly from construction. For (ii), however, one must prove that if $n \in \mathcal{N}$ and $n' \in \mathcal{N}^* \setminus \mathcal{N}$ with $t_{n,n'} \geq 1$, then $\alpha_{n,n'} > 1$. This is proved in [3] Proposition 3.3.11. □

3.4 Association of cycles and polytopes

In this section we will describe two methods of associating a cycle to a function. On the other hand, we will associate a Newton polytope to any cycle which will allow us to use the geometry of the Newton diagram to prove properties of the computation sequences defined in section 5.

3.4.1 Definition. Let $g \in \mathcal{O}_{\mathbb{C}^3,0}$ and denote by $\bar{g}$ the corresponding element in $\mathcal{O}_{X,0} = \mathcal{O}_{\mathbb{C}^3,0}/(f)$.

- ❀ For any $v \in \mathcal{V}^*$ let $\mathrm{wt}_v(g) = \min_{p\in\mathrm{supp}(g)} \ell_v(p)$ if $g \neq 0$, otherwise set $\mathrm{wt}_v(g) = \infty$. Further, let $\mathrm{wt}(g) = \sum_{v\in\mathcal{V}} \mathrm{wt}_v(g) E_v \in L$.
- ❀ For any $v \in \mathcal{V}^*$ let $\mathrm{wt}_v(\bar{g}) = \max_{h\in\mathcal{O}_{\mathbb{C}^3,0}} \mathrm{wt}_v(g+hf)$ if $\bar{g} \neq 0$, otherwise let $\mathrm{wt}_v(\bar{g}) = \infty$. Further, let $\mathrm{wt}(\bar{g}) = \sum_{v\in\mathcal{V}} \mathrm{wt}_v(\bar{g}) E_v \in L$.
- ❀ For any $v \in \mathcal{V}$, let div_v be the divisorial valuation associated with the exceptional divisor E_v. Further, let $\mathrm{div}(g) = \mathrm{div}(\bar{g}) = \sum_{v\in\mathcal{V}} \mathrm{div}_v(g) E_v \in L$.

3.4.2 Remark. To any $v \in \mathcal{V}$ there corresponds a component, say D_v, of the exceptional divisor of the modification of $\mathbb{C}^3$ inducing the resolution of X. Then wt_v is the divisorial valuation on $\mathcal{O}_{\mathbb{C}^3,0}$ associated with D_v. However, wt and div are generally not the same on $\mathcal{O}_{X,0}$, see 6.1.2.

3.4.3 Definition. Let $Z \in L$ and $v \in \mathcal{V}$. Start by defining the hyperplane and halfspace

$$\begin{aligned} H_v^{=}(Z) &= \left\{ p \in \mathbb{R}^3 \,\middle|\, \ell_v = m_v(Z) \right\} \\ H_v^{\geq}(Z) &= \left\{ p \in \mathbb{R}^3 \,\middle|\, \ell_v \geq m_v(Z) \right\}. \end{aligned}$$

Since $H_v^{=}(Z) = \left\{ p \in \mathbb{R}^3 \,\middle|\, \ell_v = m_v(Z) \right\}$ only depends on the number $m = m_v(Z)$, we also set $H_v^{=}(m) = H_v^{=}(Z)$ and $H_v^{\geq}(m) = H_v^{\geq}(Z)$. We define the *Newton polytope* of Z as

$$\Gamma_+(Z) = \mathbb{R}^3_{\geq 0} \cap \bigcap_{v \in \mathcal{V}} H_v^{\geq}(Z).$$

The *face* corresponding to a node $n \in \mathcal{N}$ is

$$F_n(Z) = \Gamma_+(Z) \cap H_n^{=}(Z).$$

The *polygon* corresponding to $n \in \mathcal{N}$ is

$$F_n^{\mathrm{nb}}(Z) = H_v^{=}(Z) \cap \bigcap_{u \in \mathcal{V}_n} H_u^{\geq}(Z).$$

3.4.4 Remark. (i) Note that for any $Z \in L$, the Newton polytope $\Gamma_+(Z)$ and its faces $F_v(Z)$ are, by definition, subsets of the positive octant $\mathbb{R}^3_{\geq 0}$. The polygons $F_v^{\mathrm{nb}}(Z)$, however, may contain points with negative coordinates.

(ii) By remark 3.2.6(i) we have $\mathcal{V}_n = \mathcal{V}_n^*$ for any $n \in \mathcal{N}$. Therefore, $F_n^{\mathrm{nb}}(Z)$ is always a finite polygon (or empty).

3.4.5. We finish this subsection by a well known formula for the anticanonical cycle Z_K.

3.4.6 Proposition (Merle and Teissier [29] 2.1.1, Oka [56] 9.1)**.** *We have* $Z_K - E = \mathrm{wt}(f) - \mathrm{wt}(x_1x_2x_3)$. □

3.4.7 Corollary. *We have* $\Gamma_+(Z_K - E) = (\Gamma_+(f) - (1,1,1)) \cap \mathbb{R}^3_{\geq 0}$.

3.5 The Newton filtration

Given a convenient $f \in \mathcal{O}_{\mathbb{C}^3,0}$ Kouchnirenko defines a filtration on $\mathcal{O}_{\mathbb{C}^3,0}$ called the *Newton filtration* [17]. In this subsection we provide an equivalent definition.

3.5.1 Definition. Let $\Gamma(f) \subset \mathbb{R}^3$ be the Newton diagram of a function $f \in \mathcal{O}_{\mathbb{C}^3}$. Recall the functions ℓ_n for $n \in \mathcal{N}$ defined in subsection 3.2. For $p \in \mathbb{N}^3$, we set $\ell_f(p) = \min_{n \in \mathcal{N}} \ell_n(p)/\,\mathrm{wt}_n(f)$. For any $0 \neq g \in \mathcal{O}_{\mathbb{C}^3,0}$ representing $\overline{g} \in \mathcal{O}_{X,0}$, we set

$$\varphi(g) = \min_{p \in \mathrm{supp}(g)} \ell_f(p), \quad \varphi(\overline{g}) = \max_{h \in (f)} \varphi(g+h).$$

We refer to both ℓ_f an φ as the *Newton weight function.* This yields the *Newton filtrations* on $\mathcal{O}_{\mathbb{C}^3,0}$ and $\mathcal{O}_{X,0}$

$$\mathcal{A}_{\mathbb{C}^3}(r) = \left\{ g \in \mathcal{O}_{\mathbb{C}^3,0} \,\middle|\, \varphi(g) \geq r \right\}, \quad \mathcal{A}_X(r) = \{ \overline{g} \in \mathcal{O}_{X,0} \,|\, \varphi(\overline{g}) \geq r \}$$

for $r \in \mathbb{Q}$ and the associated graded rings

$$\begin{aligned} A_{\mathbb{C}^3} &= \oplus_r A_{\mathbb{C}^3,r}, \quad A_{\mathbb{C}^3,r} = \mathcal{A}_{\mathbb{C}^3}(r)/\cup_{s<r}\mathcal{A}_{\mathbb{C}^3}(s), \\ A_X &= \oplus_r A_{X,r}, \quad A_{X,r} = \mathcal{A}_X(r)/\cup_{s<r}\mathcal{A}_X(s). \end{aligned}$$

The associated *Poincaré series* are given as

$$P^{\mathcal{A}}_{\mathbb{C}^3}(t) = \sum_{r\in\mathbb{Q}} \dim_{\mathbb{C}} A_{\mathbb{C}^3,r} t^r, \quad P^{\mathcal{A}}_X(t) = \sum_{r\in\mathbb{Q}} \dim_{\mathbb{C}} A_{X,r} t^r.$$

3.5.2 Remark. It follows that there is an $M \in \mathbb{Z}$ so that if $A_{\mathbb{C}^3,r} \neq 0$ then $r \in \frac{1}{M}\mathbb{N}$. In [17], the Newton filtration is normalized in such a way that for r big, we have $A_{\mathbb{C}^3,r} \neq 0$ if and only if $r \in \mathbb{N}$. For the rest of this subsection, we will fix this M and use it in proofs.

3.5.3 Lemma. *We have* $P^{\mathcal{A}}_{\mathbb{C}^3}(t) = \sum_{p\in\mathbb{N}^3} t^{\varphi(x^p)}$ *and* $P^{\mathcal{A}}_X(t) = (1-t)P^{\mathcal{A}}_{\mathbb{C}^3}(t)$.

Proof. Note that for any $g \in \mathcal{O}_{\mathbb{C}^3,0}$ we have $\varphi(fg) = \varphi(g) + 1$. Thus, for any $r \in \mathbb{Q}$ we have a sequence

$$0 \longrightarrow A_{\mathbb{C}^3,r-1} \xrightarrow{\cdot f} A_{\mathbb{C}^3,r} \longrightarrow A_{X,r} \longrightarrow 0$$

whose exactness proves the statement. The rows of the diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{A}_{\mathbb{C}^3}\left(r-1+\frac{1}{M}\right) \arrow[r] \arrow[d] & \mathcal{A}_{\mathbb{C}^3}(r-1) \arrow[r] \arrow[d] & A_{\mathbb{C}^3,r-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}_{\mathbb{C}^3}\left(r+\frac{1}{M}\right) \arrow[r] \arrow[d] & \mathcal{A}_{\mathbb{C}^3}(r) \arrow[r] \arrow[d] & A_{\mathbb{C}^3,r} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}_X\left(r+\frac{1}{M}\right) \arrow[r] \arrow[d] & \mathcal{A}_X(r) \arrow[r] \arrow[d] & A_{X,r} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

are exact and the columns are complexes. Furthermore, the first two columns are exact. The exactness of the third column now follows from the long exact sequence. □

3.5.4 Theorem. *Identify the spectrum (see subsection 2.11) with its image under the canonical isomorphism* $\mathbb{Z}[t^{\pm 1/\infty}] \cong \mathbb{Z}[\mathbb{Q}]$. *The polynomial part of* *the Poincaré series associated with the Newton filtration then recovers the part* $\mathrm{Sp}_{\leq 0}(f,0)$ *of the spectrum via the formula*

$$\mathrm{Sp}_{\leq 0}(f,0) = P_X^{\mathcal{A},\mathrm{pol}}(t^{-1}).$$

Proof. By proposition 2.11.9, it is enough to prove

$$P_X^{\mathcal{A},\mathrm{pol}}(t) = \sum_{p\in\mathbb{Z}^3_{>0}\cap\Gamma_-(f)} t^{1-\varphi(x^p)}. \tag{3.2}$$

First assume that $\Gamma(f)$ is convenient. Take an integral triangulation of $\Gamma(f)$. This can be achieved by taking any triangular face of $\Gamma(f)$ as a triangle, but subdividing any trapezoid into two triangles. Let (σ) be the family of cells in this triangulation which are not contained in any coordinate hyperplane. Thus, σ is either a triangle, or the one dimensional intersection of two adjacent triangles. For each σ let $P_\sigma(t) = \sum_{p\in\sigma\cap\mathbb{Z}^3} t^{\ell_f(p)}$ and similarly for any segment τ. Then, similarly as in [17], using lemma 3.5.3, we find

$$P_{\mathbb{C}^3}^{\mathcal{A}}(t) = \sum_\sigma (-1)^{\dim\sigma} P_\sigma(t) \tag{3.3}$$

hence

$$P_X^{\mathcal{A}}(t) = (1-t)\left(\sum_\sigma (-1)^{\dim\sigma} P_\sigma(t)\right). \tag{3.4}$$

Now, the function ℓ_f takes constant value 1 on any σ, thus lemma 2.10.6 immediately gives eq. 3.2.

Assume now that $\Gamma(f)$ is not convenient, say, $\Gamma_-(f)$ does not intersect the x_1 axis. Define the series $\tilde{P}(t)$ as the right hand side of eq. 3.3. Using the proof above, we see that $((1-t)\tilde{P}(t))^{\mathrm{pol}}$ is the right hand side of eq. 3.2. It is therefore enough to prove $P_X^{\mathcal{A},\mathrm{pol}}(t) = ((1-t)\tilde{P}(t))^{\mathrm{pol}}$

For simplicity, let us assume that $\Gamma_-(f)$ intersects the x_2 and x_3 axis. Then, modulo permutation of the last two coordinates, there is an edge in $\partial\Gamma(f)$ of the form $[q,r]$ where $q=(q_1,0,q_2)$ and $p=(r_1,1,0)$ with $q_1+r_1>0$. We find

$$P_{\mathbb{C}^3,0}^{\mathcal{A}}(t) - \tilde{P}(t) = \sum_{p\in C\cap\mathbb{Z}^3} t^{\varphi(x^p)} - \sum_{p\in C'\cap\mathbb{Z}^3} t^{\varphi(x^p)}$$

where $C = \mathbb{R}_{\geq 0}\langle (1,0,0), q, r\rangle$ and $C' = \mathbb{R}_{\geq 0}\langle q,r\rangle$. It is now enough to prove

$$\left((1-t)\sum_{p\in C\cap\mathbb{Z}^3} t^{\varphi(x^p)}\right)^{\mathrm{pol}} = 0, \quad \left((1-t)\sum_{p\in C'\cap\mathbb{Z}^3} t^{\varphi(x^p)}\right)^{\mathrm{pol}} = 0. \tag{3.5}$$

For the first one, note that as abstract semigroups, we have $C\cap\mathbb{Z}^3 = (C''\cap\mathbb{Z}^3)\oplus\mathbb{N}\langle r\rangle$, where $C'' = \mathbb{R}_{\geq 0}\langle q,(1,0,0)\rangle$, and $\varphi(x^r)=1$. Therefore, we have.

$$(1-t)\sum_{p\in C\cap\mathbb{Z}^3} t^{\varphi(x^p)} = \sum_{p\in C''\cap\mathbb{Z}^3} t^{\varphi(x^p)}.$$

The result therefore follows from lemma 2.10.6. The second equality in eq. 3.5 also follows from lemma 2.10.6, since the half parallelogram $S = \{p\in (C')^\circ \,|\, \varphi(x^p)\geq 1\}$ contains no integral points. This is because for $p=(p_1,p_2,p_3)\in S$ we have $0<p_2<1$. □

3.6 The anatomy of Newton diagrams

In this subsection we will recall some classification results of Braun and Némethi [3] which will serve as basis for the case-by-case analysis in section 7. We will also fix some notation.

3.6.1 Definition. Let $n \in \mathcal{N}$. A *leg* of n is a sequence of vertices $v_1, \ldots, v_s \in V$ so that for $j = 1, \ldots, s-1$ we have $\mathcal{V}_{v_j} = \{v_{j-1}, v_{j+1}\}$ where we set $v_0 = n$ and $\delta_{v_s} = 1$. In this case, v_s is called the *end* of the leg. The set of all ends of legs of n is denoted by $\mathcal{E}_n$, and we set $\mathcal{E} = \cup_{n\in\mathcal{N}}\mathcal{E}_n$. If $e \in \mathcal{E}$, then there are unique $n_e \in \mathcal{N}$ and $n_e^* \in \mathcal{N}^* \setminus \mathcal{N}$ so that $e \in \mathcal{E}_n$ and e lies on the bamboo connecting n_e and n_e^*. For $e = v_s \in \mathcal{E}$ as above, define $\alpha_e/\beta_e = [b_{v_1}, \ldots, b_{v_s}]$ as the *fraction* of e, where $\alpha_e, \beta_e \in \mathbb{N}$ and $\gcd(\alpha_e, \beta_e) = 1$. Thus, $\alpha_e = \alpha_{n_e,n_e^*}$ and $\beta_e = \beta_{n_e,n_e^*}$. Define also $u_e = u_{n_e,n_e^*}$. A *leg group* is a maximal nonempty set of legs for which the ratio α_e/β_e is fixed, where e is the end of the leg.

3.6.2 Definition.

- A two dimensional triangular face of $\Gamma(f)$ is called a *central triangle* if it intersects all three coordinate hyperplanes, but none of the coordinate axis. The corresponding node is called a *central node.*
- A *trapezoid* in $\Gamma(f)$ is a face whose vertices (modulo permutation) are of the form $(0,p,a)$, $(q,0,a)$, $(r_1,r_2,0)$, $(r_1',r_2',0)$ where $(r_1',r_2',0)-(r_1,r_2,0) = k(-q,p,0)$ for some $k>0$.
- An edge in Γ (a one dimensional face) is called a *central edge* if it intersects all three coordinate hyperplanes.

A *central face* is a central triangle or a trapezoid. The corresponding node is a *central node*. .

3.6.3 Definition. The collection of faces of $\Gamma(f)$ (of positive dimension) whose vertices lie on the union of two of the coordinate hyperplanes is called an *arm.* If the intersection of the two planes is the x_i axis, then we say that the arm goes in the direction of the x_i axis. An arm is *degenerate* if it does not contain a two dimensional face.

3.6.4 Proposition (Braun and Némethi [3] Proposition 2.3.9). *Let $f \in \mathcal{O}_{\mathbb{C}^3,0}$ be a function germ with Newton nondegenerate principal part, defining an isolated singularity $(X,0)$ with a rational homology sphere link. Then exactly one of the following hold (see 3.2.3 for definition of $t_{n,n'}$):*

(i) $\Gamma(f)$ has a central face and three (possibly degenerate) arms. We have $\mathcal{N} = \cup_{\kappa=1}^3\{n_0^\kappa, \ldots, n_{j^\kappa}^\kappa\}$ where $n_0^1 = n_0^2 = n_0^3$ is the central face and the arm in the direction of the x_κ axis is $F_{n_1^{(\kappa)}}\cup\ldots\cup F_{n_{j^{(\kappa)}}^{(\kappa)}}$ in the nondegenerate case, or the corresponding edge of F_{n_0} in the degenerate case. We have $t_{n_r^\kappa,n_{r'}^{\kappa'}} = 1$ if $\{r,r'\} = \{0,1\}$ or $\kappa = \kappa'$ and $|r-r'| = 1$ and $t_{n_r^\kappa,n_{r'}^{\kappa'}} = 0$ otherwise (recall definiton of $t_{n,n'}$ in 3.2.3).

(ii) There exist $c > 0$ central edges. We have $\mathcal{N} = \cup_{\kappa=1}^2\{n_1^\kappa, \ldots, n_{j^\kappa}^\kappa\}$ with $n_r^1 = n_{c-r}^2$ if $1 \le r \le c-1$. Further, we have $t_{n_r^\kappa,n_{r'}^{\kappa'}} = 1$ if $\kappa = \kappa'$ and $|r-r'| = 1$ or $\kappa \neq \kappa'$ and $|r-(c-r')| = 1$ and $t_{n_r^\kappa,n_{r'}^{\kappa'}} = 0$ otherwise.

In case (ii), if $j_\kappa \ge c$, we set $n_0^{\kappa'} = n_c^\kappa$, where $\{\kappa,\kappa'\} = \{1,2\}$.

3.6.5 Proposition. *Let $n_r = n_r^\kappa$, $r = 1, \dots, j = j^\kappa$ be an arm as in proposition 3.6.4, (i) or (ii). Assume that the arm goes in the direction of x_3.*

(i) For any $1 \le r < j$, the numbers α_e, β_e are independent of the choice of $e \in \mathcal{E}_{n_r}$. Furthermore, we have either $\ell_{n_e^*} = \ell_1$ for all $e \in \mathcal{E}_{n_r}$, or $\ell_{n_e^*} = \ell_2$ for all $e \in \mathcal{E}_{n_r}$. That is, n_r has a unique leg group.

(ii) There are two distinct integral functions $\tilde\ell_1, \tilde\ell_2 : \mathbb{R}^3 \to \mathbb{R}$ so that $\{\ell_{n_e^*} \,|\, e \in \mathcal{E}_{n_j}\} = \{\tilde\ell_1, \tilde\ell_2\}$. After possibly permuting the coordinates x_1, x_2, we have $\tilde\ell_1 = \ell_1$ and either $\tilde\ell_2 = \ell_2$, or there is an $a \in \mathbb{Z}_{\ge 0}$ so that $\tilde\ell_2 = a\ell_2 + \ell_1$.

(iii) With $\tilde\ell_1, \tilde\ell_2$ as above, set $\mathcal{E}_{n_j}^\lambda = \left\{ e \in \mathcal{E}_{n_j} \,\middle|\, \ell_{n_\lambda^*} = \tilde\ell_e \right\}$ for $\lambda = 1, 2$. We then have integers $\alpha_\lambda, \beta_\lambda$ for $\lambda = 1, 2$ so that $\alpha_e = \alpha_\lambda$ and $\beta_e = \beta_\lambda$ for $e \in \mathcal{E}_{n_j}^\lambda$. That is, n_r has exactly two leg groups. Furthermore, if $\tilde\ell_2 = \ell_2$, then $\gcd(\alpha_1, \alpha_2) = 1$, but if $\tilde\ell_2 = a\ell_2 + \ell_1$, then $\alpha_1 | \alpha_2$.

4 Two dimensional real affine geometry

In this section we describe some technical results about polygons in affine spaces. If $H \subset \mathbb{R}^3$ is a hyperplane given by an affine equation with integral coefficients so that $H \cap \mathbb{Z}^3 \neq \emptyset$, then there exists an affine isomorphism $H \to \mathbb{R}^2$, restricting to an isomorphism $H \cap \mathbb{Z}^3 \to \mathbb{Z}^2$. When dealing with such hyperplanes in $\mathbb{R}^3$, we implicitly assume such an identification given, which allows us to apply results obtained in $\mathbb{R}^2$.

4.1 General theory and classification

4.1.1 Definition. An *integral polygon* F is the convex hull $\mathrm{conv}(P)$ of a finite set of integral points spanning $\mathbb{R}^2$ as an affine space. A *vertex* of F is an element $p \in P$ so that $\mathrm{conv}(P \setminus \{p\}) \neq F$. An *edge* of F is a segment contained in the boundary of F whose endpoints are vertices.

4.1.2 Definition. A *regular vertex* p of F is a vertex having the property that primitive vectors parallel to the two boundary segments having p as an endpoint form an integral basis of $\mathbb{R}^2$. A vertex which is not regular is called *singular*.

The boundary ∂F and open kernel F° of a polygon F will have their usual meaning and an *internal point* of F is nothing but an element of F°. By an *integral affine isomorphism* we mean an $\mathbb{R}$-affine automorphism $\mathbb{R}^2 \to \mathbb{R}^2$ restricting to a $\mathbb{Z}$-affine automorphism $\mathbb{Z}^2 \to \mathbb{Z}^2$.

4.1.3 Definition. An integral polygon $F \subset \mathbb{R}^2$ is *empty* if $F^\circ \cap \mathbb{Z}^2 = \emptyset$.

4.1.4 Example. If $F_n \subset \Gamma(f)$ as above where f has Newton nondegenerate principal part and defines an isolated singularity with a rational homology sphere link, then F_n is an empty polygon in the hyperplane $H_n^=(\mathrm{wt}(f))$.

4.1.5 Proposition. *Let $F \subset \mathbb{R}^2$ be an empty integral polygon. Then, after, perhaps, applying an integral affine isomorphism on $\mathbb{R}^2$, one of the following holds.*

- Big triangle: *We have $F = \mathrm{conv}\{(0,0), (2,0), (0,2)\}$.*

❀ t-triangle: *We have* $F = \operatorname{conv}\{(0,0),(t,0),(0,1)\}$ *for some* $t \geq 0$.

❀ t-trapezoid: *We have* $F = \operatorname{conv}\{(0,0),(t,0),(0,1),(1,1)\}$ *for some* $t \geq 0$.

❀ t,s-trapezoid: *We have* $F = \operatorname{conv}\{(0,0),(t,0),(0,1),(s,1)\}$ *for some* $t \geq s > 1$.

□

4.1.6 Definition. Given a t-trapezoid F as above, with $t > 1$, the edge $[(0,1),(1,1)]$ is called the *top edge*. This edge can be identified independently of coordinates as the unique edge of length one, whose adjacend edges both have lenth one.

4.1.7 Corollary. *If $F \subset \mathbb{R}^2$ is an empty polygon and $p \in F$ is a singular vertex, then F is a t-triangle with $t > 1$, and assuming F is of the form given in proposition 4.1.5, we have $p = (0,1)$. Equivalently, F is a triangle and the opposing edge to p is not primitive.* □

4.1.8 Example. (i) An exercise shows that the only Newton diagrams as in example 4.1.4 containing big triangles are $\Gamma(x_1^{2a} + x_2^{2b} + x_3^{2c})$ where a, b, c are pairwise coprime positive integers.

(ii) Similarly, A Newton diagram as in example 4.1.4 can not contain a t,s-trapezoid. In fact, in [3], Braun and Némethi show that such a diagram can contain at most one t-trapezoid.

4.1.9 Definition. Let $F \in \mathbb{R}^2$ be an integral polygon and $S \subset F$ an edge. The unique primitive integral affine function $\ell_S : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\ell_S|_S \equiv 0$ and $\ell_S|_F \geq 0$ is called the *support function* of S with respect to F.

More generally, if $r \in \mathbb{R}_+$, then we have the diluted polygon rF which is not necessarily integral, but the term *edge* retains its meaning. The support function of an edge $S \subset rF$ is the unique primitive integral affine function $\ell_S : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\ell_S|_S \equiv m_S \in]-1,0]$ and $\ell_S|_{rF} \geq m_S$.

4.1.10 Lemma. *Let $p, q, r \in F$ be vertices of an empty polygon so that the segments $[p,q]$ and $[q,r]$ are edges of F, $[p,q]$ is primitive and q is regular. Then $\ell_{[q,r]}(p) = 1$.*

Proof. This follows more or less from definition. □

4.2 Counting lattice points in diluted polygons

4.2.1 Definition. Let F be an empty integral polygon with an edge $S = [p,q] \subset \partial F$. The *content* c_S of S is the content of the vector $q - p$.

4.2.2 Theorem. *Let $F \subset \mathbb{R}^2$ be an empty integral polygon and $r \in \mathbb{R}_+$. Furthermore, for any edge $S \subset \partial F$ with $\ell_{rS}|_{rS} \equiv 0$, choose $\varepsilon_S \in \{0,1\}$, for other edges let $\varepsilon_S = 0$. Let $F^- = F \setminus \cup_{\varepsilon_S = 1} S$. Then, there is a number $c_{rF^-} \in \mathbb{Z}$ satisfying $\sum_{S \subset \partial F} c_S(\ell_{rS} - \varepsilon_S) \equiv c_{rF^-}$ and*

$$\max\{0, c_{rF^-} + 1\} = \begin{cases} |rF^- \cap \mathbb{Z}^2| & \text{if } r < 1, \\ |rF^- \cap \mathbb{Z}^2| - |(r-1)F^- \cap \mathbb{Z}^2| & \text{if } r \geq 1. \end{cases}$$

4.2.3 Remark. If we consider $\mathbb{R}^2$ as an abstract affine plane only (with the affine lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$), then the number c_{rF^-} above depends on the polygon rF^- and cannot be determined from F^- and r alone unless one fixes an origin.

4.2.4 Definition. We call the number c_{rF^-} in the theorem above the *content* of the diluted polygon rF^- with boundary conditions.

Proof of theorem 4.2.2. We start by showing that the sum $\sum_{S\subset F} c_S(\ell_{rS}-\varepsilon_S)$ is a constant function. Since the epsilons are already constant, it is enough to show that $\sum_{S\subset F} c_S\ell_{rS}$ is constant, i.e., assume $\varepsilon_S = 0$ for all edges S. Furthermore, for any S, the difference $\ell_S - \ell_{rS}$ is a constant (since the segments S and rS are parallel), so we may assume that $r = 1$. In the case when F is a 1-triangle, we have sides S_1, S_2, S_3 and a simple exercise shows that $\ell_{S_1} + \ell_{S_2} + \ell_{S_3} = 1$, hence, $\sum_S c_S\ell_S \equiv 1$. If F is any integral polygon, take an integral triangulation of F, that is, write $F = \cup_k F_k$ where F_k are 1-triangles, and $\dim(F_k \cap F_h) \leq 1$ for $k \neq h$. We then get $\sum_{S\subset\partial F} c_S\ell_S = \sum_k \sum_{S\subset\partial F_k} c_S\ell_S$ which is a constant by the above result. Here, we have equality because in the second sum, if $S = F_k \cap F_h$, then ℓ_S is counted twice, with opposite sign and if $S \subset \partial F$ is a primitive boundary segment, then ℓ_S is counted once.

We define c_{rF-} as the value of this constant function. We will prove the theorem in the cases of a t-triangle or a t-trapezoid. One proves the theorem in the cases of a big triangle or a t,s-trapezoid using similar methods.

We start with the case when F is a t-triangle, and $\varepsilon_S = 0$ for all edges S. Write $\partial F = S_1 \cup S_2 \cup S_3$, where the S_k are edges and S_1 has length t, thus S_2 and S_3 have length 1. If $r \geq 1$, then $(r-1)F^- + p \subset rF^-$ and we have

$$|rF^- \cap \mathbb{Z}^2| - |(r-1)F^- \cap \mathbb{Z}^2| = |rF^- \setminus ((r-1)F^- + p) \cap \mathbb{Z}^2|.$$

Furthermore, we have

$$rF^- \setminus ((r-1)F^- + p) \cap \mathbb{Z}^2 = \left\{p \in rF^- \cap \mathbb{Z}^2 \,\middle|\, \ell_{rS_1}(p) = 0\right\}. \tag{4.1}$$

Note that in the case when $r < 1$, the set $rF^- \cap \mathbb{Z}^2$ is also given by the right hand side above. Therefore, to prove the lemma, we must prove

$$\max\{0, c_{rF^-} + 1\} = \left|\left\{p \in rF^- \cap \mathbb{Z}^2 \,\middle|\, \ell_{rS_1}(p) = 0\right\}\right|. \tag{4.2}$$

Since the endpoints of the segment S_1 are both regular vertices, the support functions ℓ_{rS_2}, ℓ_{rS_3} restrict to primitive functions $\ell_{rS_2}|_{L_1}, \ell_{rS_3}|_{L_1} : L_1 \to \mathbb{R}$, where $L_1 = \left\{p \in \mathbb{R}^2 \,\middle|\, \ell_{rS_1}(p) = 0\right\}$. Therefore, if the right hand side of eq. 4.2 is nonempty, then it is given as $\{p_0, \ldots, p_c\}$ where $\ell_{rS_2}(p_k) = k$ and $\ell_{rS_3}(p_k) = c - k$. The result therefore follows by evaluating the sum $\sum_{S\subset\partial F} c_S\ell_{rS}$ at the point p_0. If the set is empty, then there is a unique point $p_0 \in L_1$ with $\ell_{rS_2}(p_0) = 0$, and we must have $\ell_{rS_2}(p_0) < 0$, hence the result.

Now, assuming that $\varepsilon_{S_1} = 0$ and $\varepsilon_{S_2} = 1$ or $\varepsilon_{S_3} = 1$, then the right hand side of eq. 4.1 is given as $\{p_{\varepsilon_{S_2}}, \ldots, p_{c-\varepsilon_{S_3}}\}$ and the result is verified in the same way. If $\varepsilon_{S_1} = 1$, then, instead of eq. 4.1, we have

$$rF^- \setminus ((r-1)F^- + p) \cap \mathbb{Z}^2 = \left\{p \in rF^- \cap \mathbb{Z}^2 \,\middle|\, \ell_1(p) = 1\right\}. \tag{4.3}$$

If this set is not empty, then it is given as $\{p_{\varepsilon_{S_2}}, \ldots, p_{c-\varepsilon_{S_3}}\}$ where $\ell_{rS_2}(p_k) = k$ and $\ell_{rS_3}(p_k) = c - k$, hence $(t(\ell_{rS_1} - 1) + (\ell_{rS_2} - \varepsilon_{S_2}) + (\ell_{rS_3} - \varepsilon_{S_3}))(p_{\varepsilon_{S_2}}) =$

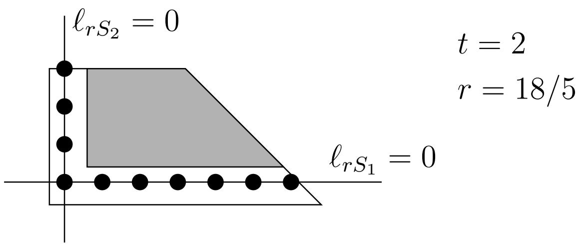

Figure 4: Counting points in $rF^- \setminus ((r-1)F^- + p)$ when F is a trapezoid.

$\ell_{rS_3}(p_0) - \varepsilon_{S_3} = c - \varepsilon_{S_2} - \varepsilon_{S_3} = |\{p_{\varepsilon_{S_2}}, \ldots, p_{c-\varepsilon_{S_3}}\}| - 1$. The result follows in a similar way as above if eq. 4.3 is empty.

The lemma is proved using a similar method if F is a t-trapezoid. Assuming this, write $\partial F = S_1 \cup S_2 \cup S_3 \cup S_4$, where the edge S_1 has length t, and S_k and S_{k+1} intersect in a vertex. If $r \geq 1$, then

$$|rF^- \cap \mathbb{Z}^2| - |(r-1)F^- \cap \mathbb{Z}^2| = |rF^- \setminus ((r-1)F^- + p) \cap \mathbb{Z}^2|.$$

where p is the intersection point of S_3 and S_4. We get

$$rF^- \setminus ((r-1)F^- + p) \cap \mathbb{Z}^2 = \left\{ p \in rF^- \cap \mathbb{Z}^2 \,\middle|\, \ell_{rS_1}(p) = \varepsilon_{S_1} \text{ or } \ell_{rS_2}(p) = \varepsilon_{S_2} \right\}. \tag{4.4}$$

We see then that the right hand side above is given as $\{p_{\varepsilon_{S_2}}, \ldots, p_{c-\varepsilon_{S_4}}\} \cup \{p'_{\varepsilon_{S_1}}, \ldots, p'_{c'-\varepsilon_{S_3}}\}$, where $p_{\varepsilon_{S_2}} = p'_{\varepsilon_{S_1}}$ and

$$\begin{array}{rclcrcl} \ell_{rS_1}(p_k) & = & \varepsilon_{S_1}, & \qquad & \ell_{rS_2}(p'_k) & = & \varepsilon_{S_2}, \\ \ell_{rS_2}(p_k) & = & k, & & \ell_{rS_1}(p'_k) & = & k, \\ \ell_{rS_4}(p_k) & = & c - k, & & \ell_{rS_3}(p'_k) & = & c' - k. \end{array}$$

In particular, if we set $q = p_{\varepsilon_{S_2}} = p'_{\varepsilon_{S_1}}$, we get

$$\begin{array}{rclcrcl} \ell_{rS_1}(q) & = & \varepsilon_{S_1}, & \qquad & \ell_{rS_2}(q) & = & \varepsilon_{S_2}, \\ \ell_{rS_3}(q) & = & c' - \varepsilon_{S_1}, & & \ell_{rS_4}(q) & = & c - \varepsilon_{S_2}. \end{array}$$

This gives

$$\begin{aligned} \sum_{S \subset \partial F} c_S(\ell_S - \varepsilon_S)(q) &= t(\varepsilon_{S_1} - \varepsilon_{S_1}) + (\varepsilon_{S_2} - \varepsilon_{S_2}) + (c' - \varepsilon_{S_1} - \varepsilon_{S_3}) + (c - \varepsilon_{S_2} - \varepsilon_{S_4}) \\ &= c - \varepsilon_{S_2} - \varepsilon_{S_4} + c' - \varepsilon_{S_1} - \varepsilon_{S_3} + 1. \end{aligned}$$

The right hand side above is the cardinality of the right hand side of eq. 4.4, if this set is nonempty, otherwise it is nonpositive. This finishes the proof. $\square$

5 Construction of sequences

In this section we will construct computation sequences for certain cycles on the resolution graph of Newton nodegenerate surface singularities described in subsection 3.2 and compare the intersection numbers on the right hand side of eq. 2.5 with a lattice point count "under the diagram". In section 6 we will use these results to identify the geometric genus topologically and in section 7 we make the same identification of the normalized Seiberg–Witten invariant of the canonical spin^c structure.

In subsection 5.1 we give a technical result which essentially allows us to work in a reduced lattice. These ideas are already present in [35, 20, 45] In subsection 5.2 we give an algorithm, which explicitly constructs the computation sequences which we will consider. In subsection 5.3 we compute some intersection numbers coming from these computation sequences.

5.1 Laufer sequences

In this section, L is the lattice associated with a resolution graph G of a normal surfaces singularity as described in subsection 2.2. In applications of the results presented, G will be the graph constructed by Oka's algorithm in subsection 3.2. We will describe a closure operator x on L and the associated generalized Laufer sequences. Némethi considers a similar operator in [35] in a specific case, and in [19] László provides a general theory. Many of the proofs in this subsection can be found in these sources. See also [46].

5.1.1 Proposition. *Let $Z \in L$. There exists a unique cycle $x(Z)$ satisfying the following properties:*

(i) $m_n(x(Z)) = m_n(Z)$ *for all* $n \in \mathcal{N}$*.*

(ii) $(x(Z), E_v) \leq 0$ *for all* $v \in \mathcal{V} \setminus \mathcal{N}$*.*

(iii) $x(Z)$ *is minimal with respect to the above conditions.*

Proof. Let $\overline{G} = G \setminus \mathcal{N}$ be the subgraph of G generated by the vertex set $\mathcal{V} \setminus \mathcal{N}$. Finding an element $x(Z)$ satisfying the above conditions is clearly equivalent to finding a minimal element $Z_{\overline{G}}$ in the lattice $L_{\overline{G}}$ associated with $\overline{G}$ satisfying

(ii') For all $v \in \mathcal{V} \setminus \mathcal{N}$ we have $(Z_{\overline{G}}, E_v)_{\overline{G}} \leq -\sum_{n \in \mathcal{N} \cap \mathcal{V}_v} m_n$.

The existence of a minimal element satisfying (ii'), as well as its uniqueness, now follows in a similar way as that of the minimal cycle, see definition 2.3.5 □

5.1.2 Remark. The above proposition and its proof hold if we replace $\mathcal{N}$ with any subset of $\mathcal{V}$.

5.1.3 Proposition. *If $Z \leq x(Z)$ then $x(Z)$ can be calculated using a computation sequence (often referred as a* generalized Laufer sequence *) as follows. Start by setting $Z_0 = Z$. Then, assuming that Z_i has been defined, if we have $(Z_i, E_v) \leq 0$ for all $v \in \mathcal{V} \setminus \mathcal{N}$, then $Z_i = x(Z)$. Otherwise, there is a $v(i)$ so that $(Z_i, E_{v(i)}) > 0$ and we define $Z_{i+1} = Z_i + E_{v(i)}$.*

Proof. It is enough to prove the following: If $Z \leq x(Z)$ and $v \in \mathcal{V} \setminus \mathcal{N}$ so that $(Z, E_v) > 0$, then $Z + E_v \leq x(Z)$. Indeed, assuming the contrary, we have $m_v(Z) = m_v(x(Z))$, hence $(Z, E_v) = (x(Z), E_v) - (x(Z) - Z, E_v) \leq 0$, a contradiction. □

5.1.4 Proposition. *The operator x satisfies the following properties:*

(i) If $Z_1, Z_2 \in L$ and $m_n(Z_1) \leq m_n(Z_2)$ for all $n \in \mathcal{N}$ then $x(Z_1) \leq x(Z_2)$.

(ii) $x(x(Z)) = x(Z)$ *for all* $Z \in L$.

(iii) *Let* $Z \in L$ *and* $Z' \in L_{\mathbb{Q}}$ *and assume that* $m_n(Z) = m_n(Z')$ *for all* $n \in \mathcal{N}$ *and* $(Z', E_v) = 0$ *for all* $v \in \mathcal{V} \setminus \mathcal{N}$. *Then* $x(Z) \geq Z'$, *with equality if* $Z' \in L$.

Proof. For (i), define $Z' \in L$ by $m_n(Z') = m_n(Z_1)$ for $n \in \mathcal{N}$ and $m_v(Z') = m_v(x(Z_2))$ for $v \in \mathcal{V} \setminus \mathcal{N}$. Then Z' satisfies the first two conditions in proposition 5.1.1 for $Z = Z_1$. By definition, we get $x(Z_1) \leq Z' \leq x(Z_2)$.

(ii) follows immediately from definition.

For (iii), let $\overline{G} = G \setminus \mathcal{N}$. Assume that $Z_1 \in L$ satisfies (i) and (ii) of proposition 5.1.1. Write $Z_1 = Z' + Z'_1$ where $\mathrm{supp}(Z'_1) \cap \mathcal{N} = \emptyset$. Then, we have $(Z'_1, E_v) \leq 0$ for all $v \in \mathcal{V} \setminus \mathcal{N}$. Applying lemma 2.2.12 to each connected component of $\overline{G}$ we find $Z'_1 \geq 0$. If $Z' \in L$, then $Z'_1 \in L$ and by minimality, $Z'_1 = 0$. $\square$

5.1.5 Lemma. *Let* $Z \in L$ *and take* $n \in \mathcal{N}$ *and* $n' \in \mathcal{N}^*_n$. *Let* $u \in \mathcal{V}_n$ *be the neighbour of* n *in the connected component of* $G \setminus n$ *containing* n'. *If* $Z = x(Z)$, *then*

$$m_u(Z) = \left\lceil \frac{\beta_{n,n'} m_n(Z) + m_{n'}(Z)}{\alpha_{n,n'}} \right\rceil, \tag{5.1}$$

where we set $m_{n'}(Z) = 0$ *if* $n' \in \mathcal{N}^* \setminus \mathcal{N}$.

Proof. Let $v_1, \dots, v_s$ be the vertices of the bamboo between n and n' as in fig. 1. We will assume that $s \geq 2$, since, in the cases $s = 0$ or $s = 1$, the lemma is a simple consequence of the definition. Set also $v_0 = n$ and $v_{s+1} = n'$. The condition $Z = x(Z)$ then implies that the sequence $(m_r)_{r=0}^{s+1}$, given by $m_r = m_{v_r}(Z)$ is the minimal family satisfying $m_0 = m_n(Z)$, $m_{s+1} = m_{n'}(Z)$ and $m_{r-1} - b_{v_r} m_r + m_{r+1} \leq 0$ for $1 \leq r \leq s$. Let $m'_0 = m_0$ and

$$m'_{s+1} = \inf \{ m \in \mathbb{Z} \,|\, m \geq m_{s+1},\ \beta_{n,n'} m_0 + m \equiv 0 \,(\mathrm{mod}\, \alpha_{n,n'}) \}.$$

Since $\beta_{n,n'} m_0 + m_{s+1} \equiv 0 \,(\mathrm{mod}\, \alpha_{n,n'})$, the equations

$$\begin{array}{rcrcrcrl} & - & b_1 m'_1 & + & m'_2 & = & -m'_0 & \\ m'_{r-1} & - & b_1 m'_r & + & m'_{r+1} & = & 0 & \quad 1 < r < s \\ m'_{s-1} & - & b_1 m'_s & & & = & -m'_{s+1} & \end{array} \tag{5.2}$$

have integral solutions $m'_1, \dots, m'_s$, see e.g. [61] or [2], III.5. Furthermore, we have

$$m'_1 = \frac{\beta_{n,n'} m'_0(Z) + m'_{s+1}(Z)}{\alpha_{n,n'}} = \left\lceil \frac{\beta_{n,n'} m_n(Z) + m_{n'}(Z)}{\alpha_{n,n'}} \right\rceil.$$

By proposition 5.1.4(iii), $m'_1, \dots, m'_s$ is the minimal sequence satisfying $m'_{r-1} - b_r m'_r + m'_{r+1} \leq 0$ for all $1 \leq r \leq s$, and by proposition 5.1.4(i) we have $m_1 \leq m'_1$. Thus, we have proved the $\leq$ part of eq. 5.1.

For the opposite inequality, set $m''_0 = m_0$, $m''_{s+1} = m_{s+1}$ and take $m''_1, \dots, m''_s$ as the rational solution of eq. 5.2, with m'_0 and m'_{s+1} on the right side replaced with m''_0 and m''_{s+1}. Then we have $m_{r-1} - b_r m_r + m_{r+1}$ for $0 < r < s+1$, and so $m_r \geq m''_r$ for all r by proposition 5.1.4(i). But we also have

$$m''_1 = \frac{\beta_{n,n'} m_0(Z) + m_{s+1}(Z)}{\alpha_{n,n'}},$$

hence, the $\geq$ part of eq. 5.1. $\square$

5.1.6 Lemma. *Let G be a graph constructed by Oka's algorithm as in subsection 3.2 from the Newton diagram $\Gamma(f)$. We have $x(0) = 0$. Furthermore,*

(i) If G is the graph of a minimal good resolution (i.e. $\Gamma(f)$ is a minimal diagram) then $x(Z_K) = Z_K$.

(ii) If $\Gamma(f)$ is convenient, then $x(\mathrm{wt}(f)) = \mathrm{wt}(f)$.

(iii) Without the assumption of minimality or convenience, we have $x(Z_K - E) = Z_K - E + Z_{\mathrm{legs}}$ where Z_{legs} is the support of all legs in G (see definition 3.6.1).

Proof. The equality $x(0) = 0$ follows from proposition 5.1.4(iii). Similarly, (ii) follows from the same lemma, once we show that if $v \in \mathcal{V}\setminus\mathcal{N}$, then $(\mathrm{wt}(f), E_v) = 0$. For such a v, there are $n \in \mathcal{N}$ and $n' \in \mathcal{N}_n^*$ so that v is on a bamboo connecting n and n' as in fig. 1. Set $v_0 = n$ and $v_{s+1} = n'$. We then get $\mathrm{wt}_{v_r}(f) = \ell_{v_r}(p)$ for $0 \leq r \leq s$, as well as $\ell_{v_{s+1}}(p) = 0$ (this follows from convenience). We therefore get $(\mathrm{wt}(f), E_{v_r}) = \ell_{v_{r-1}}(p) - b_{v_r}\ell_{v_r}(p) + \ell_{v_{r+1}}(p) = 0$ for $1 \leq r < s$, and $(\mathrm{wt}(f), E_{v_s}) = \ell_{v_{s-1}}(p) - b_{v_s}\ell_{v_s}(p) = \ell_{v_{s-1}}(p) - b_{v_s}\ell_{v_s}(p) + \ell_{v_{r+1}}(p) = 0$.

Next, we prove (iii). We start with showing $Z_K - E \leq x(Z_K - E)$. By negative definiteness, there exists a rational solution $(m_v)_{v\in\mathcal{V}\setminus\mathcal{N}}$ to the linear equations $-b_v m_v + \sum_{u\in\mathcal{V}_v\setminus\mathcal{N}} m_v = -\sum_{n\in\mathcal{N}\cap\mathcal{V}_v} m_n(Z_K - E)$ for $v \in \mathcal{V}\setminus\mathcal{N}$. Take $Z \in L_{\mathbb{Q}}$ with $m_n(Z) = m_n(Z_K - E)$ for $n \in \mathcal{N}$ and $m_v(Z) = m_v$ for $v \in \mathcal{V}\setminus\mathcal{N}$ and set $Z_1 = Z_K - E - Z$. Then Z_1 is supported on $\mathcal{V}\setminus\mathcal{N}$ and we have $(Z_1, E_v) = (Z_K - E, E_v) \geq 0$ for $v \in \mathcal{V}\setminus\mathcal{N}$. By lemma 2.2.12, we have $Z_1 \leq 0$, thus $Z_K - E \leq Z \leq x(Z_K - E)$ by proposition 5.1.4(iii). Now, if $e \in \mathcal{E}$, we have $(Z_K - E, E_e) = 1$, and so we can start a computation sequence as in proposition 5.1.3 with e. Using the notation $v_1, \ldots, v_s$ as in definition 3.6.1, we show that if we already have a computation sequence $v_s, v_{s-1}, \ldots, v_{r-1}$ for some $r > 1$, we may take v_r as the next element. But this follows from the fact that $(Z_K - E - E_{v_s} - \ldots - E_{v_{r-1}}, E_r) = 1$. Thus, we get a computation sequence starting with $Z_K - E$, ending with $Z_K - E + Z_{\mathrm{legs}}$, at which point we have $(Z_K - E + Z_{\mathrm{legs}}, E_v) \leq 0$ for all $v \in \mathcal{V}\setminus\mathcal{N}$. Indeed, if $v \in \mathcal{V}\setminus\mathcal{N}$ is not on a leg, then $(Z_K - E + Z_{\mathrm{legs}}, E_v) = (Z_K - E, E_v) = \delta_v - 2 = 0$. If $v = v_1$ with the notation above, then we get $(Z_K - E + Z_{\mathrm{legs}}, E_v) = (Z_K, E_v) - 1 = -b_v + 1 \leq 0$ and if $v = v_r$ with $r > 1$, we get $(Z_K - E + Z_{\mathrm{legs}}, E_v) = (Z_K, E_v) = -b_v + 2 \leq 0$. This proves $x(Z_K - E) = Z_K - E + Z_{\mathrm{legs}}$.

Finally, we prove item (i). To calculate $x(Z_K)$, we can construct a computation sequence as in proposition 5.1.3 starting at $Z_K - E + Z_{\mathrm{legs}} + \sum_{n\in\mathcal{N}} E_n$, since $Z_K - E + Z_{\mathrm{legs}} = x(Z_K - E) \leq x(Z_K)$ by proposition 5.1.4(i) and the above computations, and therefore, $Z_K - E + Z_{\mathrm{legs}} + \sum_{n\in\mathcal{N}} E_n \leq x(Z_K)$. This sequence is similar to the above. Take any $n, n' \in \mathcal{N}$ with $n' \in \mathcal{N}_n$ and a bamboo $v_1, \ldots, v_s$ connecting n, n' as in fig. 1. We can then take $v_1, \ldots, v_s$ as the start of the computation sequence. This is becasue if $Z = Z_K - E + Z_{\mathrm{legs}} + \sum_{n\in\mathcal{N}} E_n + E_{v_1} + \ldots + E_{v_{r-1}}$, then $(Z, E_{v_r}) = m_{v_{r-1}}(Z_K) - b_{v_1}(m_{v_r}(Z_K) - 1) + m_{v_{r+1}}(Z_K) - 1 = (Z_K, E_{v_r}) + b_{v_r} - 1 = 1$ by the adjunction equalities. Now, the concatenation of all the sequences along such bamboos gives a sequence which ends at Z_K. Furthermore, we have $(Z_K, E_v) \leq 0$ for all $v \in \mathcal{V}\setminus\mathcal{N}$ by the minimality assumption, so this is where the sequence stops. $\square$

5.2 Algorithms

In this subsection we give three different constructions for a computation sequence, each having some good properties.

5.2.1 Definition. The *ratio test* is a choice of a node $n \in \mathcal{N}$ given a cycle $Z \in L$. More precisely, we consider the following three minimising conditions:

I. Given $Z \in L$, choose n to minimise the fraction
$$\frac{m_n(Z)}{m_n(Z_K - E)}.$$

II. Given $Z \in L$, choose n to minimise the fraction
$$\frac{m_n(Z)}{\mathrm{wt}_n(f)}.$$

III. Given $Z \in L$, choose n to minimise the fraction
$$\frac{m_n(Z) + \mathrm{wt}_n(x_1x_2x_3)}{\mathrm{wt}_n(f)}.$$

If given a choice between more than one nodes minimizing the given fraction, we choose one maximising the intersection number (Z, E_n). We also define $Z^I = Z_K$, $Z^{II} = \mathrm{wt}(f)$ and $Z^{III} = x(Z_K - E)$.

5.2.2 Definition. Computation sequence $* = $ I,II,III is defined recursively as follows. Start by setting $\bar{Z}_0 = 0$. Given $\bar{Z}_i$, if $\bar{Z}_i = Z^*$, then stop the algorithm. Otherwise, choose $\bar{v}(i) \in \mathcal{N}$ according to ratio test $*$ and set $\bar{Z}_{i+1} = x(\bar{Z}_i + E_{\bar{v}(i)})$. We obtain a computation sequence (Z_i) for Z^* by connecting $\bar{Z}_i + E_{\bar{v}(i)}$ and $\bar{Z}_{i+1}$ using the generalized Laufer sequence from proposition 5.1.3. This is possible since we have $\bar{Z}_i = x(\bar{Z}_i) \leq x(\bar{Z}_i + E_{v(i)}) = \bar{Z}_{i+1}$ by proposition 5.1.4, and so $\bar{Z}_i + E_{v(i)} \leq x(\bar{Z}_i + E_{\bar{v}(i)})$. Note also that by lemma 5.1.6 and convention 5.2.3, we have $x(Z^*) = Z^*$ in each case.

In case I, we will only consider the finite sequence going from 0 to Z_K. In case and III, similarly, we will only consider the finite sequence going from 0 to $x(Z_K - E)$. In the case II we continue the sequence to infinity, as in definition 2.8.1, yielding an infinite sequence $(\bar{Z}_i)_{i=0}^{\infty}$.

5.2.3 Convention. In case I, we will assume that the diagram $\Gamma(f)$ is minimal, whereas in cases II and III, we will assume that the diagram is convenient. This is motivated by the following facts. Ratio test I can be made for any nonrational minimal graph, yielding a computation sequence to Z_K in the numerically Gorenstein case. By proposition 3.3.3, the minimal resolution graph is obtained by Oka's algorithm, assuming that $\Gamma(f)$ is a minimal diagram. Therefore, although we use our knowledge of the diagram $\Gamma(f)$ in the proofs of our statements, the statements themselves can be made entirely in terms of the link M. In particular, the geometric genus can be computed using only the link.

In cases II and III, we already assume the knowledge of $\mathrm{wt}(x_1x_2x_3)$ in order to construct the computation sequence. Given a diagram $\Gamma(f)$ of an arbitrary function $f \in \mathcal{O}_{\mathbb{C}^3,0}$ with Newton nondegenerate principal part, defining an isolated singularity with rational homology sphere link, let $f' = f + \sum_{c=1}^{3} x_c^d$, where

$d \in \mathbb{N}$ is large. Then f and f' define analytically equivalent germs. Furthermore, let G' be the graph obtained from Oka's algorithm run on the diagram $\Gamma(f')$. For any $e \in \mathcal{E}$, set $\gamma_e = -(Z_K - E + \mathrm{wt}(f), E_e) + 1$ if $(Z_K - E + \mathrm{wt}(f), E_e) \neq 0$, but $\gamma_e = 0$ otherwise. Assuming a good choice of d, the graph G' is then obtained from the graph G by blowing up each end γ_e times. Therefore, assuming that $\Gamma(f)$ is convenient imposes no restriction in generality if we already assume the knowledge of $\mathrm{wt}(x_1x_2x_3)$.

5.2.4 Remark. (i) The number k will be fixed throughout as the number of steps in the sequence $(\bar{Z}_i)_i$. However, it depends on which case we are following. In order not to complicate the notation, this is not indicated. In case I we have $k = \sum_{n\in\mathcal{N}} m_n(Z_K)$, in case III we have $k = \sum_{n\in\mathcal{N}} m_n(Z_K - E)$ and in case II, we have $k = \sum_{n\in\mathcal{N}} \mathrm{wt}_n(f)$.
(ii) Note that $(\bar{Z}_i)$ forms a subsequence of (Z_i) as in remark 2.8.4. From the viewpoint of theorem 2.8.2, the only interesting part of the computation sequences constructed in definition 5.2.2 are the terms $\bar{Z}_i$.

5.3 Intersection numbers and lattice point count

In this subsection we assume that we have constructed a computation sequence $(\bar{Z}_i)_{i=0}^k$ as in the previous subsection. The main result is theorem 5.3.2 which connects numerical data obtained from the sequence $(\bar{Z}_i)_{i=0}^k$ with a lattice point count associated with the Newton diagram.

We remark that in this section, and in what follows, in cases II and III, we assume that the Newton diagram $\Gamma(f)$ is convenient. In case I, however, we assume that $\Gamma(f)$ is minimal.

5.3.1 Definition. In cases I, II, III, for any i, define

$$a_i = \max\{0, (-Z_i, E_{v(i)}) + 1\}.$$

Furthermore, set $P_i = (\Gamma_+(Z_i) \setminus \Gamma_+(Z_{i+1})) \cap \mathbb{Z}^3$. Set also $\bar{a}_i = a_{i'}$ and $\bar{P}_i = P_{i'}$ if $\bar{Z}_i = Z_{i'}$. Thus, in cases I, III we have a sequence $(\bar{a})_{i=0}^{k-1}$, whereas in case II we consider the infinite sequence $(\bar{a})_{i=0}^{\infty}$.

5.3.2 Theorem. *Assume the notation introduced above and in subsection 3.4 as well as the sequence $(\bar{Z}_i)_{i=0}^k$ defined in definition 5.2.2. In case II, consider as well the continuation of the sequence as in definition 2.8.1. Then the following hold:*

(i) In cases I, III we have $\mathbb{Z}^3_{\geq 0} \setminus \Gamma_+(Z_K - E) = \amalg_{i=0}^{k-1} \bar{P}_i$ and $|\bar{P}_i| = \bar{a}_i$ for all $i = 0, \ldots, k-1$.

(ii) In case II we have $\mathbb{Z}^3_{\geq 0} = \amalg_{i=0}^{\infty} \bar{P}_i$. In particular, $\mathbb{Z}^3_{\geq 0} \setminus \Gamma_+(\mathrm{wt}(f)) = \amalg_{i=0}^{k} \bar{P}_i$. Furthermore, we have $|\bar{P}_i| = \bar{a}_i$ if $i < k$ and $|\bar{P}_i| - |\bar{P}_{i-k}| = \bar{a}_i$ if $i \geq k$.

In order to simplify the proof of theorem 5.3.2, we start with some lemmas. The proof of the theorem is given in the end of the section.

5.3.3 Definition. To each node $n \in \mathcal{N}$ in the graph we associate a cone C_n and at each step in the algorithm we record the minimal fraction from the ratio test.

- In case I we set

$$C_n = \mathbb{R}_{\geq 0} F_n(Z_K - E), \quad \bar{r}_i = \frac{m_{v(i)}(\bar{Z}_i)}{m_{v(i)}(Z_K - E)}.$$

  Furthermore, for any $n \in \mathcal{N}$, set $\varepsilon_{i,n} = 1$ if $m_n(\bar{Z}_i) = \bar{r}_i m_n(Z_K - E) + 1$, but $\varepsilon_{i,n} = 0$ otherwise.

- In case II we set

$$C_n = \mathbb{R}_{\geq 0} F_n, \quad \bar{r}_i = \frac{m_{v(i)}(\bar{Z}_i)}{\mathrm{wt}_{v(i)}(f)}.$$

  Furthermore, for any $n \in \mathcal{N}$, set $\varepsilon_{i,n} = 1$ if $m_n(\bar{Z}_i) = \bar{r}_i \,\mathrm{wt}_n(f) + 1$, but $\varepsilon_{i,n} = 0$ otherwise.

- In case III we set

$$C_n = (\mathbb{R}_{\geq 0} F_n - (1,1,1)) \cap \mathbb{R}^3_{>-1}, \quad \bar{r}_i = \frac{m_{v(i)}(\bar{Z}_i) + \mathrm{wt}_{v(i)}(x_1x_2x_3)}{\mathrm{wt}_{v(i)}(f)}.$$

  Furthermore, for any $n \in \mathcal{N}$, we set $\varepsilon_{i,n} = 1$ if $m_n(\bar{Z}_i) + \mathrm{wt}_n(x_1x_2x_3) = \bar{r}_i \,\mathrm{wt}_n(f) + 1$, but $\varepsilon_{i,n} = 0$ otherwise.

Fix a step i of the computation sequence in cases I, II, III. For $n \in \mathcal{N}_{\bar{v}(i)}$, take $u = u_{\bar{v}(i),n} \in \mathcal{V}_{\bar{v}(i)}$ and define $\varepsilon_{i,u} = 1$ if $\varepsilon_{i,n} = 1$ and $\beta_{\bar{v}(i),n} m_{\bar{v}(i)}(\bar{Z}_i) + m_n(\bar{Z}_i) - 1 \equiv 0 \,(\mathrm{mod}\, \alpha_{\bar{v}(i),n})$, otherwise, set $\varepsilon_{i,u} = 0$. For $n \in \mathcal{N}^*_{\bar{v}(i)} \setminus \mathcal{N}$, we use the following definition.

- In case I, set $\varepsilon_{i,u} = 1$ if $\bar{r}_i = 1$, but $\varepsilon_{i,u} = 0$ otherwise.

- In case II, set $\varepsilon_{i,u} = 0$ for all i.

- In case III, set $\varepsilon_{i,u} = 1$ if $m_{\bar{v}(i)}(\bar{Z}_i) + \mathrm{wt}_{\bar{v}(i)}(x_1x_2x_3) - 1 \equiv 0 \,(\mathrm{mod}\, \alpha_{n,n'})$, but $\varepsilon_{i,u} = 0$ otherwise.

Although in case III, the sets C_n are not technically cones, we still refer to them as such.

5.3.4 Remark. It can happen that for an $n \in \mathcal{N}_{\bar{v}(i)}$ and $u = u_{\bar{v}(i),n}$ we have $n = u$. In this case, $\alpha_{\bar{v}(i),n} = 1$, so the condition $\beta_{\bar{v}(i),n} m_{\bar{v}(i)}(\bar{Z}_i) + m_n(\bar{Z}_i) - 1 \equiv 0 \,(\mathrm{mod}\, \alpha_{\bar{v}(i),n})$ is vacuous. Therefore, $\varepsilon_{i,u} = \varepsilon_{i,n}$ is well defined.

5.3.5 Lemma. *For any $i \geq 0$ and $n \in \mathcal{N}$ we have*

$$m_n(\bar{Z}_i) = \begin{cases} \lceil \bar{r}_i m_n(Z_K - E) + \varepsilon_{i,n} \rceil & \textit{in case I}, \\ \lceil \bar{r}_i \,\mathrm{wt}_n(f) + \varepsilon_{i,n} \rceil & \textit{in case II}, \\ \lceil \bar{r}_i \,\mathrm{wt}_n(f) - \mathrm{wt}_n(x_1x_2x_3) + \varepsilon_{i,n} \rceil & \textit{in case III}. \end{cases} \tag{5.3}$$

Similarly, if $\bar{v}(i) = n$ and $u \in \mathcal{V}_n$, then

$$m_u(\bar{Z}_i) = \begin{cases} \lceil \bar{r}_i m_u(Z_K - E) + \varepsilon_{i,u} \rceil & \textit{in case I}, \\ \lceil \bar{r}_i \,\mathrm{wt}_u(f) + \varepsilon_{i,u} \rceil & \textit{in case II}, \\ \lceil \bar{r}_i \,\mathrm{wt}_u(f) - \mathrm{wt}_u(x_1x_2x_3) + \varepsilon_{i,u} \rceil & \textit{in case III}. \end{cases} \tag{5.4}$$

Proof. We prove eq. 5.3 in case I, the other cases are similar. For a fixed i and $n \in \mathcal{N}$, set $i' = \max\{a \in \mathbb{N} \,|\, a \leq i,\, \bar{v}(a) = n\}$. If $n = \bar{v}(i)$, then the statement is clear, so we will assume that $i \neq i'$. Then $m_n(\bar{Z}_i) = m_n(\bar{Z}_{i'}) + 1$. The ratio test guarantees that the sequence $(\bar{r}_i)$ is increasing. In particular, we have $\bar{r}_{i'} \leq \bar{r}_i$, hence

$$\frac{m_n(\bar{Z}_{i'})}{m_n(Z_K - E)} = \bar{r}_{i'} \leq \bar{r}_i$$

and so $m_n(\bar{Z}_i) - 1 = m_n(\bar{Z}_{i'}) \leq \bar{r}_i m_n(Z_K - E)$. The ratio test furthermore gives $\bar{r}_i m_n(Z_K - E) \leq m_n(\bar{Z}_i)$. Therefore, we have

$$\bar{r}_i m_n(Z_K - E) \leq m_n(\bar{Z}_i) \leq \bar{r}_i m_n(Z_K - E) + 1.$$

If we have equality in the second inequality above, then $\varepsilon_{i,n} = 1$ and the result holds. Otherwise, we have $\varepsilon_{i,n} = 0$ and $m_n(\bar{Z}_i) = \lceil \bar{r}_i m_n(Z_K - E) \rceil$, which also proves the result.

Next, we prove eq. 5.4 in case I, the other cases follow similarly. Assume first that $n = \bar{v}(i)$ for some i, and that $u = u_{n,n'}$ for some $n' \in \mathcal{N}_n$. If $\varepsilon_{i,u} = 1$, then we get, by lemma 5.1.5 and the definition of $\varepsilon_{i,u}$ and the above result,

$$\begin{aligned}
m_u(\bar{Z}_i) &= \left\lceil \frac{\beta_{n,n'} m_n(\bar{Z}_i) + m_{n'}(\bar{Z}_i)}{\alpha_{n,n'}} \right\rceil \\
&= \frac{\beta_{n,n'} m_n(\bar{Z}_i) + m_{n'}(\bar{Z}_i) - 1}{\alpha_{n,n'}} + 1 \\
&= \bar{r}_i \frac{\beta_{n,n'} m_n(Z_K - E) + m_{n'}(Z_K - E)}{\alpha_{n,n'}} + 1 \\
&= \bar{r}_i m_u(Z_K - E) + 1.
\end{aligned}$$

The result follows by a similar string of equalities in the case $\varepsilon_{i,n'} = 1 \neq \varepsilon_{i,u}$ as in the case $\varepsilon_{i,n'} = 0 = \varepsilon_{i,u}$. If, on the other hand, $n' \in \mathcal{N}_n^* \setminus \mathcal{N}$, then

$$\begin{aligned}
m_u(\bar{Z}_i) &= \left\lceil \frac{\beta_{n,n'} m_n(\bar{Z}_i)}{\alpha_{n,n'}} \right\rceil \\
&= \left\lceil \frac{\beta_{n,n'} \bar{r}_i m_n(Z_K - E)}{\alpha_{n,n'}} \right\rceil \\
&= \left\lceil \bar{r}_i \frac{\alpha_{n,n'} m_u(Z_K - E) + 1}{\alpha_{n,n'}} \right\rceil \\
&= \begin{cases} \lceil \bar{r}_i m_u(Z_K - E) \rceil & \bar{r}_i < 1, \\ \lceil \bar{r}_i m_u(Z_K - E) \rceil + 1 & \bar{r}_i = 1. \end{cases}
\end{aligned}$$

Here, the first equalities follow as before. The case $\bar{r}_i = 1$ is clear. The inequality $\bar{r}_i < 1$ is equivalent to $m_n(\bar{Z}_i) < m_n(Z_K - E)$. Assuming this, we must prove

$$\left\lceil \frac{m_n(\bar{Z}_i)\alpha_{n,n'} m_u(Z_K - E) + m_n(\bar{Z}_i)}{m_n(Z_K - E)\alpha_{n,n'}} \right\rceil = \left\lceil \frac{m_n(\bar{Z}_i)\alpha_{n,n'} m_u(Z_K - E)}{m_n(Z_K - E)\alpha_{n,n'}} \right\rceil.$$

In order to prove the above equation, we will show that there is no integer $k \in \mathbb{Z}$ satisfying

$$m_n(\bar{Z}_i)\alpha_{n,n'} m_u(Z_K - E) + m_n(\bar{Z}_i) \geq m_n(Z_K - E)\alpha_{n,n'} k > m_n(\bar{Z}_i)\alpha_{n,n'} m_u(Z_K - E).$$

Using lemma 3.2.8, this is equivalent to

$$m_n(\bar{Z}_i)\beta_{n,n'}m_n(Z_K-E) \geq m_n(Z_K-E)\alpha_{n,n'}k > m_n(\bar{Z}_i)(\beta_{n,n'}m_n(Z_K-E)-1)$$

i.e.

$$\beta_{n,n'}m_n(Z_K-E) \geq \alpha_{n,n'}k > \beta_{n,n'}m_n(Z_K-E) - \bar{r}_i.$$

But this is impossible by the assumption $0 \leq \bar{r}_i < 1$. □

5.3.6 Lemma. *Let $Z \in L$ and assume that $(Z, E_v) > 0$ for some $v \in \mathcal{V}$. Then $F_v^{\mathrm{nb}}(Z) \cap \mathbb{R}^3_{\geq 0} = \emptyset = F_v(Z)$. If, furthermore, $v \in \mathcal{N}$, then $F_v^{\mathrm{nb}}(Z) = \emptyset$.*

Proof. Assuming that there is a point $p \in F_v^{\mathrm{nb}}(Z) \cap \mathbb{R}^3_{\geq 0}$ we arrive at the following contradiction

$$0 < -b_v m_v(Z) + \sum_{u \in \mathcal{V}_v} m_u(Z) \leq -b_v \ell_v(p) + \sum_{u \in \mathcal{V}_v} \ell_u(p) = - \sum_{u \in \mathcal{V}_v^* \setminus \mathcal{V}} \ell_u(p) \leq 0$$

where the equality is eq. 3.1. The last inequality follows since $\ell_v(p) \geq 0$ for all $v \in \mathcal{V}^*$ and $p \in \mathbb{R}^3_{\geq 0}$. Furthermore, we have $F_v(Z) \subset F_v^{\mathrm{nb}}(Z)$. The second statement follows in the same way, since, by construction, we have $\mathcal{V}_v = \mathcal{V}_v^*$ if $v \in \mathcal{N}$. □

5.3.7 Lemma. *The cones C_n, for $n \in \mathcal{N}$, are given as follows:*

(i) In case I

$$C_n = \left\{ p \in \mathbb{R}^3 \;\middle|\; \forall n' \in \mathcal{N}_n^* : \frac{\ell_{n'}(p)}{m_{n'}(Z_K - E)} \geq \frac{\ell_n(p)}{m_n(Z_K - E)} \right\}$$

where we replace $m_{n'}(Z_K - E)$ with -1 if $n' \in \mathcal{N}^* \setminus \mathcal{N}$.

(ii) In case II

$$C_n = \left\{ p \in \mathbb{R}^3 \;\middle|\; \forall n' \in \mathcal{N}_n : \frac{\ell_{n'}(p)}{\mathrm{wt}_{n'}(f)} \geq \frac{\ell_n(p)}{\mathrm{wt}_n(f)}, \quad \forall n' \in \mathcal{N}_n^* \setminus \mathcal{N} : \ell_{n'}(p) \geq 0 \right\}.$$

(iii) In case III

$$C_n = \left\{ p \in \mathbb{R}^3_{>-1} \;\middle|\; \begin{array}{l} \forall n' \in \mathcal{N}_n : \dfrac{\ell_{n'}(p) + \mathrm{wt}_{n'}(x_1x_2x_3)}{\mathrm{wt}_{n'}(f)} \geq \dfrac{\ell_n(p) + \mathrm{wt}_n(x_1x_2x_3)}{\mathrm{wt}_n(f)}, \\ \forall n' \in \mathcal{N}_n^* \setminus \mathcal{N} : \ell_{n'}(p) > -1 \end{array} \right\}.$$

Proof. The face $F_n - (1,1,1)$ is given by the equation $\ell_n = m_n(Z_K - E)$ and the inequalities $\ell_{n'} \geq m_{n'}(Z_K - E)$ for $n' \in \mathcal{N}_n$. (i) therefore follows, since C_n is the cone over $F_n - (1,1,1)$.

For (ii), we have, similarly as above, that C_n is given by inequalities $\ell_{n'}/\mathrm{wt}_{n'}(f) \geq \ell_n/\mathrm{wt}_n(f)$ for $n' \in \mathcal{N}_n$ and $\ell_{n'} \geq 0$ if $n' \in \mathcal{N}_n^*$. If $n' \in \mathcal{N}_n^* \setminus \mathcal{N}$, then $\ell_{n'}$ is one of the coordinate functions. Since $F_n \subset \mathbb{R}^3_{\geq 0}$, the above inequalities are equivalent with $\ell_{n'}/\mathrm{wt}_{n'}(f) \geq \ell_n/\mathrm{wt}_n(f)$ for $n' \in \mathcal{N}_n$ and $\ell_c \geq 0$ for $c = 1,2,3$.

(iii) follows in a similar way as (ii). □

5.3.8 Lemma. *Let $n \in \mathcal{N}$. We have $F_n^{\mathrm{nb}}(\mathrm{wt}(f)) = F_n$ and $F_n^{\mathrm{nb}}(Z_K - E) = F_n - (1,1,1)$. Furthermore, $F_n^{\mathrm{nb}}(Z_K - E)$ consists of those points $p \in H_n^=(Z_K - E)$ satisfying $\ell_{n'}(p) \geq m_{n'}(Z_K - E)$ for $n' \in \mathcal{N}_n$ and $\ell_{n'}(p) \geq -1$ for $n' \in \mathcal{N}_n^* \setminus \mathcal{N}$.*

Proof. Start by observing that for $p \in H_n^{=}(\mathrm{wt}(f))$ and $n' \in \mathcal{N}_n^*$ we have $\ell_{n'}(p) \geq \mathrm{wt}_{n'}(f)$ if and only if $\ell_u(p) \geq \mathrm{wt}_u(f)$, where $u = u_{n,n'}$. Indeed, the halfplane defined by either inequality has boundary the affine hull of the segment $F_n \cap F_{n'}$ and contains F_n. By definition, the face F_n is defined by the equation $\ell_n(p) = \mathrm{wt}_n(f)$ and inequalities $\ell_{n'}(p) \geq \mathrm{wt}_{n'}(p)$ for $n' \in \mathcal{N}_n^*$. The equality $F_n = F_n^{\mathrm{nb}}(\mathrm{wt}(f))$ follows. This result, combined with proposition 3.4.6, provides $F_n^{\mathrm{nb}}(Z_K - E) = F_n - (1,1,1)$.

For the last statement, we observe as above that for $p \in H_n^{=}(Z_K - E)$, $n' \in \mathcal{N}_n$ and $u = u_{n,n'} \in \mathcal{V}_n$, the inequality $\ell_{n'}(p) \geq m_{n'}(Z_K - E)$ is equivalent with $\ell_u(p) \geq m_u(Z_K - E)$. Furthermore if $n' \in \mathcal{N}_n^* \setminus \mathcal{N}$, and $u = u_{n,n'}$, using

$$\begin{aligned} \alpha_{n,n'}\ell_u &= \beta_{n,n'}\ell_n + \ell_{n'} \\ \alpha_{n,n'}m_u(Z_K - E) &= \beta_{n,n'}m_n(Z_K - E) - 1 \end{aligned} \tag{5.5}$$

we find that $\ell_u(p) \geq m_u(Z_K - E)$ if and only if $\ell_{n'}(p) \geq -1$, since we are assuming that $\ell_n(p) = m_n(Z_K - E)$. Here, the first equality in eq. 5.5 follows from remark 3.2.2(ii) and the second one is lemma 3.2.8. $\square$

5.3.9 Definition. For any i, let $F_i^{\mathrm{cn}} = C_{\bar v(i)} \cap H_{\bar v(i)}^{=}(\bar Z_i)$. For any $u \in \mathcal{V}_{\bar v(i)}$, let $S_{i,u}$ be the minimal set of ℓ_u on F_i^{cn}, and set $F_i^{\mathrm{cn}-} = F_i^{\mathrm{cn}} \setminus \cup_{\varepsilon_{i,u}=1} S_{u,i}$.

5.3.10 Lemma. *In cases I, III, for $i = 0, \ldots, k-1$ and in case II, for $i \geq 0$, we have*

$$\bar P_i = F_{\bar v(i)}(\bar Z_i) \cap \mathbb{Z}^3 = F_{\bar v(i)}^{\mathrm{nb}}(\bar Z_i) \cap \mathbb{Z}^3 = F_i^{\mathrm{cn}-} \cap \mathbb{Z}^3.$$

Proof. We start by proving the inclusions

$$\bar P_i \subset F_{\bar v(i)}(\bar Z_i) \cap \mathbb{Z}^3 \subset F_{\bar v(i)}^{\mathrm{nb}}(\bar Z_i) \cap \mathbb{Z}^3 \subset F_i^{\mathrm{cn}-} \cap \mathbb{Z}^3. \tag{5.6}$$

For the first inclusion in eq. 5.6, note that

$$\bar P_i = \left\{ p \in \mathbb{Z}^3 \cap \Gamma(\bar Z_i) \,\middle|\, m_{\bar v(i)}(\bar Z_i) \leq \ell_{\bar v(i)}(p) < m_{\bar v(i)}(\bar Z_i) + 1 \right\}.$$

Since the function $\ell_{\bar v(i)}$ takes integral values on integral points, we may replace the two inequalities with $\ell_{\bar v(i)}(p) = m_{\bar v(i)}(Z_i)$, yielding, in fact, $\bar P_i = F_{\bar v(i)}(\bar Z_i) \cap \mathbb{Z}^3$.

The second inclusion in eq. 5.6 follows from definition.

For the third inclusion, we prove case I, cases II and III follow in a similar way. Take $p \in F_{\bar v(i)}^{\mathrm{nb}}(\bar Z_i) \cap \mathbb{Z}^3$. Clearly, we have $\ell_{\bar v(i)}(p) = m_{\bar v(i)}(p)$, thus $p \in H_{\bar v(i)}^{=}(\bar Z_i)$. We start with proving $p \in C_{\bar v(i)}$, i.e. that p satisfies the inequalities in lemma 5.3.7(i). Take $n \in \mathcal{N}_{\bar v(i)}$ and set $u = u_{\bar v(i),n}$. Then

$$\begin{aligned} \ell_n(p) &= \alpha_{\bar v(i),n}\ell_u(p) - \beta_{\bar v(i),n}\ell_{\bar v(i)}(p) \\ &\geq \alpha_{\bar v(i),n}m_u(\bar Z_i) - \beta_{\bar v(i),n}m_{\bar v(i)}(\bar Z_i) \\ &\geq \bar r_i \left(\alpha_{\bar v(i),n}m_u(Z_K - E) - \beta_{\bar v(i),n}m_{\bar v(i)}(Z_K - E)\right) \\ &= \bar r_i m_n(Z_K - E). \end{aligned}$$

By lemma 5.3.7, this gives $p \in C_{\bar v(i)}$. If $u = u_{\bar v(i),n}$ and $\varepsilon_{i,u} = 1$, then the second inequality above would be strict by lemma 5.3.5. By lemma 5.3.7, this implies $p \notin C_n$. By definition, we get $p \in F_i^{\mathrm{cn}-}$.

By lemma 5.3.12, the sets $F^{\mathrm{cn}-}_{\bar v(i)} \cap \mathbb{Z}^3$ are pairwise disjoint. Therefore, to prove equality in eq. 5.6, it is now enough to prove $\cup_{i=0}^{k-1} \bar P_i \supset \cup_{i=0}^{k-1} F^{\mathrm{cn}-}_{\bar v(i)} \cap \mathbb{Z}^3$ in cases I and III, and $\cup_{i=0}^{\infty} \bar P_i \supset \cup_{i=0}^{\infty} F^{\mathrm{cn}-}_{\bar v(i)} \cap \mathbb{Z}^3$ in case II. But this is clear, since, by construction, we have $\cup_{i=0}^{j-1} \bar P_i = \mathbb{Z}^3_{\geq 0} \setminus \Gamma(\bar Z_j)$ for any j, hence $\cup_{i=0}^{k-1} \bar P_i = \mathbb{Z}^3_{\geq 0} \setminus \Gamma(Z_K - E)$ in cases I and III, and $\cup_{i=0}^{\infty} \bar P_i = \mathbb{Z}^3_{\geq 0}$ in case II. □

5.3.11 Lemma. *In cases I and III, if $\bar r_i = 1$ then $(\bar Z_i, E_{\bar v(i)}) > 0$. Similarly, in cases I and II, if $\bar r_i = 0$, then $(\bar Z_i, E_{\bar v(i)}) > 0$ unless $i = 0$.*

Proof. We start by proving the first statement.

For each $n \in \mathcal{N}$, there is a unique i so that $\bar v(i) = n$ and $\bar r_i = 1$. Since the sequence $\bar r_0, \dots, \bar r_{k-1}$ is, by construction, increasing, we see that $\bar Z_{k-|\mathcal{N}|} = x(Z_K - E)$, that the sequence $\bar v(k - |\mathcal{N}|), \bar v(k - |\mathcal{N}| + 1), \dots, \bar v(k-1)$ contains each element in $\mathcal{N}$ exactly once and that $\bar r_i < 1$ for $i < k - |\mathcal{N}|$. Recall that by lemma 5.1.6, we have $x(Z_K - E) = Z_K - E + Z_{\mathrm{legs}}$.

If $u = u_e \in \mathcal{V}_{\bar v(j)}$ for some $k - |\mathcal{N}| \leq j \leq k-1$, then we have $m_u(\bar Z_{j'}) = m_u(Z_K)$ for $k - |\mathcal{N}| \leq j' \leq k-1$. This clearly holds for $j = k - |\mathcal{N}|$ by lemma 5.1.6, as well as for $j = k-1$. By monotonicity, proposition 5.1.4(i), the statement holds for all $k - |\mathcal{N}| \leq j' \leq k-1$. By definition of $\varepsilon_{i,n}$ we also see $\varepsilon_{j,\bar v(j')} = 1$ if and only if $j' < j$. Thus, if $k - |\mathcal{N}| \leq j, j' \leq k-1$ and $\bar v(j') \in \mathcal{N}_{\bar v(j)}$, then, by lemmas 5.1.5 and 3.2.8,

$$m_u(\bar Z_j) = \left\lceil \frac{\beta_{n,n'} m_n(Z_K - E) + m_{n'}(Z_K - E) + \varepsilon_{j,\bar v(j')}}{\alpha_{n,n'}} \right\rceil = m_u(Z_K - E) + \varepsilon_{j,\bar v(j')}$$

where $n = \bar v(j)$ and $n' = \bar v(j')$ and $u = u_{n,n'}$. Here we use lemma 3.2.8, which implies that $\beta_{n,n'} m_n(Z_K - E) + m_{n'}(Z_K - E) \equiv 0 \,(\mathrm{mod}\, \alpha_{n,n'})$. Therefore, if $k - |\mathcal{N}| \leq j \leq j'$, we get

$$\begin{aligned}(\bar Z_j, E_{\bar v(j')}) &= (Z_K - E, E_{\bar v(j')}) + |\mathcal{E}_{v(j')}| + |\{v(j'') \,|\, j'' < j\} \cap \mathcal{N}_{v(j')}| \\ &= 2 - |\{v(j'') \,|\, j'' \geq j\} \cap \mathcal{N}_{v(j')}|\end{aligned} \tag{5.7}$$

because $(Z_K - E, E_n) = 2 - \delta_n$ and $\delta_n = |\mathcal{E}_n| + |\mathcal{N}_n|$ for all $n \in \mathcal{N}$.

For $j = k - |\mathcal{N}|, \dots, k-1$, let H_j be the graph with vertex set $\bar v(j), \dots, \bar v(k-1)$ and an edge between n, n' if and only if $n' \in \mathcal{N}_n$. We will prove by induction that the graphs H_j are all trees, i.e. connected, and that if $j < k-1$, then $v(j)$ is a leaf in H_j, that is, it has exactly one neighbour in H_j.

We know already that $H_{k-|\mathcal{N}|}$ is a tree. Furthermore, removing a leaf from a tree yields another tree. Therefore, it is enough to prove that if, for some j, the graph H_j is a tree, then $\bar v(j)$ is a leaf. The ratio test says that we must indeed choose $\bar v(j)$ from the graph H_j, maximising the intersection number $(\bar Z_j, E_{\bar v(j)})$. For simplicity, identify the graph H_j with its set of vertices. Then eq. 5.7 says that for any $n \in H_j$, we have $(\bar Z_j, E_n) = 2 - |H_j \cap \mathcal{N}_n|$. This number is clearly maximized when n is a leaf of H_j. Furthermore, we have $(\bar Z_j, E_{\bar v(i)}) = 1$ for $j < k-1$ and $(\bar Z_{k-1}, E_{\bar v(k-1)}) = 2$, proving the first statement of the lemma.

We sketch the proof of the second statement. The sequence $v(0), \dots, v(|\mathcal{N}| - 1)$ contains each element of $\mathcal{N}$ exactly once. Therefore, we have $m_{\bar v(i)}(\bar Z_i) = 0$ for $i < |\mathcal{N}|$. Similarly as in the case above, one shows that if $0 < i < |\mathcal{N}|$, then $\bar v(i)$ is chosen in such a way that there is an $i' < i$ so that $\bar v(i') \in \mathcal{N}$, and hence, $m_u(\bar Z_i) > 0$ for $u = u_{\bar v(i), \bar v(i')}$, which yields $(\bar Z_i, E_{\bar v(i)}) > 0$. □

5.3.12 Lemma. *In cases I, III, let $0 \leq i' < i \leq k-1$, in case II, let $0 \leq i' < i$. Then $F_i^{\mathrm{cn}-} \cap F_{i'}^{\mathrm{cn}-} \cap \mathbb{Z}^3 = \emptyset$.*

Proof. We will assuma that we have a point $p \in F_i^{\mathrm{cn}-} \cap F_{i'}^{\mathrm{cn}-} \cap \mathbb{Z}^3$, to arrive at a contradiction. Set $n = \bar{v}(i)$ and $n' = \bar{v}(i')$.

We start with cases I, III. In these cases we will show that $n' \in \mathcal{N}_n$ and that $\varepsilon_{i,u} = 1$ where $u = u_{n,n'}$, hence, $p \notin F_i^{\mathrm{cn}-}$, a contradiction. Since $p \in C_n \cap C_{n'}$, we have, by lemma 5.3.7

$$\bar{r}_i = \frac{m_n(\bar{Z}_i)}{m_n(Z_K - E)} = \frac{\ell_n(p)}{m_n(Z_K - E)} = \frac{\ell_{n'}(p)}{m_{n'}(Z_K - E)} = \frac{m_{n'}(\bar{Z}_{i'})}{m_{n'}(Z_K - E)} = \bar{r}_{i'}$$

in case I. In case III, we have, similarly,

$$\begin{aligned}\bar{r}_i &= \frac{m_n(\bar{Z}_i) + \mathrm{wt}_n(x_1x_2x_3)}{\mathrm{wt}_n(f)} = \frac{\ell_n(p) + \mathrm{wt}_n(x_1x_2x_3)}{\mathrm{wt}_n(f)} \\ &= \frac{\ell_{n'}(p) + \mathrm{wt}_{n'}(x_1x_2x_3)}{\mathrm{wt}_{n'}(f)} = \frac{m_{n'}(\bar{Z}_{i'}) + \mathrm{wt}_{n'}(x_1x_2x_3)}{\mathrm{wt}_{n'}(f)} = \bar{r}_i.\end{aligned}$$

The ratio test guarantees that the sequence $\bar{r}_0, \bar{r}_1, \ldots$ is increasing. In particular, there is no i'' with $i' < i'' < i$ and $v(i'') = n'$. Therefore, we find $m_{n'}(\bar{Z}_i) = m_{n'}(\bar{Z}_{i'}) + 1$. By definition, we find $\varepsilon_{i,n'} = 1$.

If $\bar{r}_i \neq 0$, then define $\tilde{p} = \bar{r}_i^{-1} p$ in case I and $\tilde{p} = \bar{r}_i^{-1}(p - (1,1,1)) + (1,1,1)$ in case III. In each case, we have $\tilde{p} \in (\Gamma(f) - (1,1,1)) \cap \mathbb{R}_{\geq 0}$, as well as $\tilde{p} \in C_n \cap C_{n'}$. In particular, $\tilde{p}$ is not in the boundary of the shifted diagram $\partial\Gamma(f) - (1,1,1)$. Therefore, the intersection $F_n \cap F_{n'}$ must be one dimensional. Thus, $n' \in \mathcal{N}_n$. Furthermore, we have $m_n(\bar{Z}_i) + m_{n'}(\bar{Z}_i) - 1 = m_n(\bar{Z}_i) + m_{n'}(\bar{Z}_{i'}) = \ell_n(p) + \ell_{n'}(p) \equiv 0 \,(\mathrm{mod}\, \alpha_{n,n'})$, hence $\varepsilon_{i,u} = 1$, where $u = u_{n,n'}$. But since $\tilde{p} \in F_n \cap F_{n'} - (1,1,1)$, the point p is in the minimal set of $\ell_{n'}$ on F_i^{cn}, thus $p \notin F_i^{\mathrm{cn}-}$. This concludes the proof in cases I, III if $\bar{r}_i \neq 0$.

By construction, we can not have $\bar{r}_i = 0$ in case III, and as we saw in the proof of lemma 5.3.11, since $i > 0$, the node n has a neighbour n'' for which $m_{n''}(\bar{Z}_i) = 1$, hence $\varepsilon_{u,i} = 1$ for $u = u_{n,n''}$, finishing the proof as above.

Next, we prove the lemma in case II. For brevity, we cite some of the methods used above. For instance, we find $\bar{r}_i = \bar{r}_{i'}$ in a similar way. The case $\bar{r}_i = 0$ can also be treated in the same way as in case I, so we will assume $\bar{r}_i > 0$. Set $\tilde{p} = \bar{r}_i^{-1} p$. Then $\tilde{p} \in \Gamma(f)$. Unless $\tilde{p}$ is an integral point, we see in the same way above that $n' \in \mathcal{N}_n$ and that $\varepsilon_{i,u} = 1$ for $u = u_{n,n'}$, finishing the proof. Therefore, assume, that $\tilde{p}$ is integral. Then $\tilde{p}$ lies on one of the coordinate hyperplanes and it lies on the boundary $\partial\Gamma(f)$. It follows that we have $n_1, \ldots, n_j \in \mathcal{N}$ so that F_{n_s} for $1 \leq s \leq j$ are precisely the faces of $\Gamma(f)$ containing $\tilde{p}$ and that $n_{s'} \in \mathcal{N}_{n_s}$ if and only if $|s - s'| = 1$. There are also have numbers $i_1, \ldots, i_j \in \mathbb{N}$ so that for each s, we have $\bar{v}(i_s) = n_s$ and $m_{\bar{v}(i_s)}(\bar{Z}_{i_s}) = \ell_{\bar{v}(i_s)}(p)$. Let σ be a permutation on $1, \ldots, j$ which orders the numbers $i_1, \ldots, i_j$, that is, $i_{\sigma(1)} < \ldots < i_{\sigma(j)}$. Just like in the previous case, we see, by lemma 5.3.7, that $\bar{r}_{i_s}$ is constant for $1 \leq s \leq j$, and so for $1 \leq s, s' \leq j$ we have $m_{n_s}(\bar{Z}_{v(i_{s'})}) = \ell_{n_s}(p) + \varepsilon_{i_{s'}, n_s}$ and $\varepsilon_{i_{s'}, n_s} = 1$ if and only if $s < s'$. Furthermore, if $u = u_{n_s, n_{s \pm 1}}$ for some s, and $\varepsilon_{i_s, n_{s\pm 1}} = 1$, then we get $\varepsilon_{i_s, u} = 1$ in the same way as before.

By the assumption $p \in C_n \cap C_{n'}$, there are s, s' so that $n = n_s = \bar{v}(i_s)$ and $n' = n_{s'} = \bar{v}(i_s)$. In particular, $i_s > i_{s'} \geq i_{\sigma(1)}$. Thus, the lemma is proved, once we show that for any s with $i_{\sigma(s)} > i_{\sigma(1)}$, we have, either $i_\sigma(s+1) < i_{\sigma(s)}$

or $i_\sigma(s-1) < i_{\sigma(s)}$, because, if e.g. $i_\sigma(s+1) < i_{\sigma(s)}$ then $\varepsilon_{i,u} = 1$ where $u = u_{n_s,n_{s+1}}$, and so $p \notin F_i^{\mathrm{cn}-}$. We will prove this using the following satement. If $\sigma(s'') \geq \sigma(s)$, then

$$(\bar{Z}_{i_s}, E_{n_{s''}}) \begin{cases} \leq 0 & \text{if} \quad \varepsilon_{i_s,n_{s''+1}} = 0 \,\text{and}\, \varepsilon_{i_s,n_{s''-1}} = 0 \\ > 0 & \text{if} \quad 1 < s'' < j \,\text{and}\, \varepsilon_{i_s,n_{s''-1}} = 1 \,\text{or}\, \varepsilon_{i_s,n_{s''+1}} = 1. \end{cases} \tag{5.8}$$

Here, we exlude the condition $\varepsilon_{i_s,n_{s''+1}} = 0$ if $s'' = j$ as well as the condition $\varepsilon_{i_s,n_{s''-1}} = 0$ if $s'' = 1$, since they have no meaning.

We finish proving the lemma assuming eq. 5.8. Assume that $1 \leq s \leq j$ and $\sigma(s) > \sigma(1)$. The node $\bar{v}(i_s)$ is chosen according to the ratio test. If there is an s'' so that $1 < s'' < j$ and $\varepsilon_{i_s,n_{s''}} = 0$, then this s'' can be chosen so that either $\varepsilon_{i_s,n_{s''+1}} = 1$ or $\varepsilon_{i_s,n_{s''-1}} = 1$, and therefore $(\bar{Z}_{i_s}, E_{n_{s''}}) > 0$ by the second part of eq. 5.8. By the maximality condition in the ratio test, we find $(\bar{Z}_{i_s}, E_{\bar{v}(i_s)}) > 0$, and therefore, by the first part of eq. 5.8, either $\varepsilon_{i_s,n_{s''+1}} = 1$ or $\varepsilon_{i_s,n_{s''-1}} = 1$, that is, either $i_\sigma(s+1) < i_{\sigma(s)}$ or $i_\sigma(s-1) < i_{\sigma(s)}$.

If, however, there is no such s'', then $\varepsilon_{i_s,n_{s''}} = 1$ for $1 < s'' < j$, and so $\bar{v}(i_s) = n_1$ or $\bar{v}(i_s) = n_j$. In the fist case, we have $\varepsilon_{i_s,n_2} = 1$ and in the second case, we have $\varepsilon_{i_s,n_{j-1}} = 1$, which, in either case, finishes the proof.

We remark that if $k = \sum_{n\in\mathcal{N}} \mathrm{wt}_n(f)$, then $\bar{v}(i+k) = \bar{v}(i)$ and $\varepsilon_{i,v} = \varepsilon_{i+k,v}$ for any v where $\varepsilon_{i,v}$ is defined. It therefore suffices to prove the above statement for $i < k$, which is equivalent to $\bar{r}_i < 0$.

We will now prove eq. 5.8. For the first part, take $1 \leq s'' \leq j$ with $\sigma(s'') \geq \sigma(s)$, hence $\varepsilon_{i_s,n_{s''}} = 0$. Assume further the given condition, namely that, if $s'' > 1$, then $\varepsilon_{i_s,n_{s''-1}} = 0$ and if $s'' < j$, then $\varepsilon_{i_s,n_{s''+1}} = 0$. Since $m_{n_{s''}}(\bar{Z}_{i_s}) = \ell_{n_{s''}}(p)$, it is enough, by eq. 3.1, to show that $m_u(\bar{Z}_i) \leq \ell_u(p)$ for all $u \in \mathcal{V}^*_{n_{s''}}$. Note that since $n_{s''} \in \mathcal{N}$ we have $\mathcal{V}^*_{n_{s''}} = \mathcal{V}_{n_{s''}}$, see remark 3.2.6(i). If $u = u_{n_{s''},n_{s''\pm1}}$, then

$$\begin{aligned} m_u(\bar{Z}_{i_s}) &= \left\lceil \frac{\beta_{n_{s''},n_{s''\pm1}} m_{n_{s''}}(\bar{Z}_i) + m_{n_{s''\pm1}}(\bar{Z}_i)}{\alpha_{n_{s''},n_{s''\pm1}}} \right\rceil \\ &= \frac{\beta_{n_{s''},n_{s''\pm1}} \ell_{n_{s''}}(p) + \ell_{n_{s''\pm1}}(p)}{\alpha_{n_{s''},n_{s''\pm1}}} \\ &= \ell_u(p) \end{aligned} \tag{5.9}$$

by lemma 5.1.5. If $u \in \mathcal{V}_{n_{s''}}$ is any other neighbour, then there is an $n'' \in \mathcal{N}^*$ so that $u = u_{n_{s''},n''}$. If $n'' \in \mathcal{N}^* \setminus \mathcal{N}$, then $n'' = n^*_e$ for some $e \in \mathcal{E}_{n_{s''}}$, and

$$m_u(\bar{Z}_i) = \left\lceil \frac{\beta_e m_{n_{s''}}(\bar{Z}_{i_s})}{\alpha_e} \right\rceil = \left\lceil \frac{\beta_e \ell_{n_{s''}}(p)}{\alpha_e} \right\rceil \leq \frac{\beta_e \ell_{n_{s''}}(p) + \ell_{n''}(p)}{\alpha_e} = \ell_u(p). \tag{5.10}$$

If $n'' \in \mathcal{N}$ and n'' is not one of the nodes $n_1, \ldots, n_j$, then $p \notin C_{n''}$, in particular, $p \in C_{n_{s''}} \setminus C_{n''}$, and so by lemma 5.3.7

$$\frac{\ell_{n''}(p)}{\mathrm{wt}_{n''}(f)} > \frac{\ell_{n_{s''}}(p)}{\mathrm{wt}_{n_{s''}}(f)} = \bar{r}_{i_s}$$

which, by lemma 5.3.5, gives $\ell_{n''}(p) \geq \bar{r}_{i_s}\,\mathrm{wt}_{n''}(f) + \varepsilon_{i_s,n''} = m_{n''}(\bar{Z}_{i_s})$ because if $\varepsilon_{i_s,n''} \neq 0$, then $\bar{r}_{i_s}\,\mathrm{wt}_{n''}(f) \in \mathbb{Z}$. This yields

$$\begin{aligned} m_u(\bar{Z}_{i_s}) &= \left\lceil \frac{\beta_{n_{s''},n''} m_{n_{s''}}(\bar{Z}_i) + m_{n''}(\bar{Z}_i)}{\alpha_{n_{s''},n''}} \right\rceil \\ &\leq \frac{\beta_{n_{s''},n''} \ell_{n_{s''}}(p) + \ell_{n''}(p)}{\alpha_{n_{s''},n''}} \\ &= \ell_u(p). \end{aligned} \tag{5.11}$$

This finishes the first part of eq. 5.8.

We prove next the second part of eq. 5.8. So, assume that $1 < s'' < j$ and that $\varepsilon_{i_s,n_{s''+1}} + \varepsilon_{i_s,n_{s''-1}} > 0$. As in eq. 5.9 we find $m_u(\bar{Z}_{i_s}) = \ell_u(p) + \varepsilon_{i_s,n_{s''\pm 1}}$, if $u = u_{n_{s''},n_{s''\pm 1}}$. The result therefore follows from eq. 3.1, once we prove $m_u(\bar{Z}_{i_s}) = \ell_u(p)$ for $u \in \mathcal{V}_{n_{s''}} \setminus \{n_{s''\pm 1}\}$.

We start with the case $u = u_e$ with $e \in \mathcal{E}_{n_{s''}}$. In this case, we will show that we have, in fact, equality in eq. 5.10 (where $n'' = n_e^*$). This follows once we prove that $\ell_{n_e^*}(p) < \alpha_e$. Since the face $F_{n_{s''}}$ has at most four edges, the edge $F_{n_{s''}} \cap F_{n_e^*}$ is adjacent to at least one of the edges $F_{n_{s''}} \cap F_{n_{s''\pm 1}}$, let us assume that it is adjacent to $F_{n_{s''}} \cap F_{n_{s''+1}}$, and define $\tilde{p}_1$ as the point of intersection of the two edges. Define also $p_1 = \bar{r}_i \tilde{p}$. Then $\tilde{p}_1$ is a vertex of the face $F_{n_{s''}}$. From corollary 4.1.7, we see that this is in fact a regular vertex, and from proposition 3.2.10 we have $\ell_{n_e^*}(\tilde{p} - \tilde{p}_1) = \alpha_{n_{s''},n_e^*}$. Furthermore, since, in case II, we assume that the diagram is convenient, the function $\ell_{n_e^*}$ is one of the coordinates, and p_1 is on the corresponding coordinate hyperplane, thus $\ell_{n_e^*}(p_1) = 0$. We get $\ell_{n_e^*}(p) = \ell_{n_e^*}(p - p_1) = \bar{r}_i \alpha_e$ and $\bar{r}_i < 1$ since we are assuming case II.

For the case when $n'' \in \mathcal{N}_{n_{s''}}$, equality in eq. 5.11 is proved in a similar way. This finishes the proof of the second part of eq. 5.8, and so, the lemma is proved. □

Proof of theorem 5.3.2. Take any i, with $0 \leq i \leq k-1$ in cases I, III, and $i \geq 0$ in case II. Each edge S of the polygon F_i^{cn} is the minimal set of some ℓ_u with $u \in \mathcal{V}_{\bar{v}(i)}$. In this case, define $\varepsilon_S = \varepsilon_{i,u}$. We have $-b_{\bar{v}(i)}\ell_{\bar{v}(i)} + \sum_{u \in \mathcal{V}_{\bar{v}(i)}} \ell_u \equiv 0$. Thus, for any $p \in H^{=}_{\bar{v}(i)}(Z_i)$, we have $-b_{\bar{v}(i)} m_{\bar{v}(i)}(\bar{Z}_i) = \sum_{u \in \mathcal{V}_{\bar{v}(i)}} \ell_u(p)$. This gives

$$(-\bar{Z}_i, E_{\bar{v}(i)}) = -b_{\bar{v}(i)} m_{\bar{v}(i)}(\bar{Z}_i) - \sum_{u \in \mathcal{V}_{\bar{v}(i)}} m_u(\bar{Z}_i) = \sum_{u \in \mathcal{V}_{\bar{v}(i)}} \ell_u(p) - m_u(\bar{Z}_i). \tag{5.12}$$

If $u \in \mathcal{V}_{\bar{v}(i)}$, then there is an $n \in \mathcal{N}^*_{\bar{v}(i)}$ so that $u = u_{\bar{v}(i),n}$. Let $S \subset F_i^{\mathrm{cn}}$ be the minimal set of ℓ_u. We then have

$$\lceil \ell_u|_S \rceil = \left\{ \begin{array}{ll} \lceil \bar{r}_i m_u(Z_K - E) \rceil & \text{in case } I \\ \lceil \bar{r}_i\,\mathrm{wt}_u(f) \rceil & \text{in case } II \\ \left\lceil \dfrac{\bar{r}_i m_u(Z_K - E) + \mathrm{wt}_u(x_1 x_2 x_3)}{\mathrm{wt}_u(f)} \right\rceil & \text{in case } III \end{array} \right\} = m_u(\bar{Z}_i) - \varepsilon_{i,u}$$

by lemma 5.3.5. Furthermore, $\ell_u|_{H^{=}_{\bar{v}(i)}(\bar{Z}_i)}$ is a primitive affine function, whose minimal set on F_i^{cn} is $\bar{r}_i S$. Using notation from section 4, it follows, that $\ell_S = \ell_u - m_u(\bar{Z}_i) + \varepsilon_S$, and so, by eq. 5.12, we have $(-\bar{Z}_i, E_{\bar{v}(i)}) = c_{F_i^{\mathrm{cn}}-}$. The theorem therefore follows from theorem 4.2.2. □

6 Calculating the geometric genus and the spectrum

In this section we assume that $(X,0) \subset (\mathbb{C}^3,0)$ is an isolated singularity with rational homology sphere link, given by a function $f \in \mathcal{O}_{\mathbb{C}^3,0}$ with Newton nondegenerate principal part. Notation from previous sections is retained.

6.1 A direct identification of p_g and $\mathrm{Sp}_{\leq 0}(f,0)$

In this subsection we give a simple formula for both the geometric genus p_g and part of the spectrum, $\mathrm{Sp}_{\leq 0}(f,0)$, in terms of computation sequences I and III. Eq. 6.1 has already been proved in [46] using the same method.

6.1.1 Theorem. *Let the computation sequence $(\bar{Z}_i)_{i=0}^k$ be defined as in definition 5.2.2, cases I, III. Recall the numbers $\bar{r}_i \in [0,1]$ from definition 5.3.3. Then, the geometric genus of $(X,0)$ is given by the formula*

$$p_g = \sum_{i=0}^{k-1} \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}. \tag{6.1}$$

Furthermore, in case III we have

$$\mathrm{Sp}_{\leq 0}(f,0) = \sum_{i=0}^{k-1} \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}[\bar{r}_i] \in \mathbb{Z}[\mathbb{Q}]. \tag{6.2}$$

6.1.2 Lemma (Ebeling and Gusin-Zade [8])**.** *Let $g \in \mathcal{O}_{\mathbb{C}^3,0}$ and $n \in \mathcal{N}$. Writing $g = \sum_{p\in\mathbb{N}^3} b_p x^p$, set $g_n = \sum_{\ell_n(p)=\mathrm{wt}_n(g)} b_p x^p$. Then $\mathrm{wt}_n(g) < \mathrm{div}_n(g)$ if and only if g_n is divisible by f_n over the ring $\mathcal{O}_{\mathbb{C}^3,0}[x_1^{-1}, x_2^{-1}, x_3^{-1}]$.* □

Proof of theorem 6.1.1. We start by proving eq. 6.1. By proposition 2.5.8, we have $p_g = h_{Z_K}$. Therefore, eq. 6.1 follows from theorem 2.8.2, once we prove

$$\dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\bar{Z}_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\bar{Z}_{i+1}))} \geq \max\{0, (\bar{Z}_i, E_{\bar{v}(i)}) + 1\} \tag{6.3}$$

for all $i = 0, \ldots, k-1$. We start by noticing that for any $p \in \bar{P}_i$ we have $x^p \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\bar{Z}_i))$ (we identify a function on $(\mathbb{C}^3,0)$ with its restriction to $(X,0)$, as well as its pullback via π to $\tilde{X}$). By theorem 5.3.2(i), the right hand side of eq. 6.3 is the cardinality of $\bar{P}_i$, and so the inequality is proved once we show that the family $(x^p)_{p\in\bar{P}_i}$ is linearly independent modulo $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\bar{Z}_{i+1}))$. So, take a $\mathbb{C}$-linear combination $g = \sum_{p\in\bar{P}_i} b_p x^p$ and assume that $g \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\bar{Z}_{i+1}))$. Since $g = g_{\bar{v}(i)}$, lemma 6.1.2 says that there is an $h \in \mathcal{O}_{\mathbb{C}^3,0}[x_1^{-1}, x_2^{-1}, x_3^{-1}]$ so that $g = hf_n$. In the case when $r_i = 1$ we have $\bar{P}_i = \emptyset$. Otherwise, we have $r_i < 1$ and the support of g is contained in a translate of $r_i F_{\bar{v}(i)}$. A simple exercise shows, however, that the convex hull of the support of hf_n must contain a translate of $F_{\bar{v}(i)}$, unless $h = 0$. We have therefore shown that $g = 0$, proving the independence of $(x^p)_{p\in\bar{P}_i}$.

For eq. 6.2, we note that if $p \in \bar{P}_i$, then $p + (1,1,1) \in \mathbb{R}_{\geq 0} F_{\bar{v}(i)}$ and so $\bar{r}_i = \ell_f(p)$ (see definition 3.5.1 for ℓ_f). The family of sets $\bar{P}_i + (1,1,1)$ provides a partition of the set $\mathbb{Z}^3_{>0} \setminus \Gamma_+(f)$. Saito's result proposition 2.11.9 therefore gives

$$\mathrm{Sp}_{\leq 0}(f,0) = \sum_{i=0}^{k-1} \sum_{p\in \bar{P}_i} [\ell_f(p)] = \sum_{i=0}^{k-1} \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)} + 1\}[\bar{r}_i].$$

□

6.2 The Poincaré series of the Newton filtration and the spectrum

In this subsection, we give a formula for the Poincaré series $P_X^{\mathcal{A}}(t)$ in terms of computation sequence II. In particular, we recover $\mathrm{Sp}_{\leq 0}(f,0)$ again.

6.2.1 Theorem. *Let $(\bar{Z}_i)_{i=0}^{\infty}$ be the computation sequence defined in definition 5.2.2, case II and define*

$$P_X^{II}(t) = \sum_{i=0}^{\infty} \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\} t^{\bar{r}_i}. \tag{6.4}$$

Then $P_X^{\mathcal{A}}(t) = P_X^{II}(t)$. In particular, we have $P_X^{II}(t) \in \mathbb{C}[t]_{(t)}$ and $\mathrm{Sp}(f,0)_{\leq 0} = P_X^{II,\mathrm{pol}}(t^{-1})$.

Proof. For any i and $p \in \bar{P}_i$, we have $\ell_f(p) = \bar{r}_i$, and $\bar{r}_{i-k} = \bar{r}_i - 1$ if $i \geq k$. Moreover, the family $(\bar{P}_i)$ is a partition of $\mathbb{Z}^3_{\geq 0}$. By theorem 5.3.2(ii) and lemma 3.5.3 we get

$$P_X^{II}(t) = (1-t)\sum_{i=0}^{\infty} |\bar{P}_i| t^{\bar{r}_i} = (1-t) \sum_{p\in\mathbb{Z}^3_{\geq 0}} t^{\ell(p)} = P_X^{\mathcal{A}}(t).$$

The other statements now follow from theorem 3.5.4. □

7 Calculating the Seiberg–Witten invariant

In this section we compare the numerical data obtained in section 5 with coefficients of the counting function $Q_0(t)$ from subsection 2.4. Using proposition 2.6.9 we recover the normalized Seiberg–Witten invariant associated with the canonical spin^c structure on the link from computation sequence I from definition 5.2.2. The strategy we will follow is similar to that of the geometric genus. We do not know whether the "main identity" $Z_0 = P$ holds (see [37]). We will, however, see that computation sequence I defined in section 5 does in fact compute the normalized Seiberg–Witten invariant $\mathbf{sw}^0_M(\sigma_{\mathrm{can}}) - (Z_K^2 + |\mathcal{V}|)/8$, using the counting function $Q_0(t)$ in the same way the geometric genus was obtained using the Hilbert function $H(t)$.

In this section we assume that M, a rational homology sphere, is the link of an isolated singularity $(X,0) \subset (\mathbb{C}^3,0)$ given by a function $f \in \mathcal{O}_{\mathbb{C}^3}$ with Newton nondegenerate principal part. We will assume that G is the minimal graph representing the link. Equivalently, it is the graph obtained by Oka's algorithm in subsection 3.2, under the assumption that the diagram $\Gamma(f)$ is minimal. Furthermore, we have the series Z, Q defined in subsection 2.4. We assume that $(\bar{Z}_i)$ is computation sequence I from definition 5.2.2.

7.0.2 Theorem. *For $i = 0, \ldots, k-1$ we have*

$$q_{\bar{Z}_{i+1}} - q_{\bar{Z}_i} = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}. \tag{7.1}$$

In particular, we have

$$\mathbf{sw}^0_M(\sigma_{\mathrm{can}}) - \frac{Z_K^2 + |\mathcal{V}|}{8} = \sum_{i=0}^{k-1} \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}. \tag{7.2}$$

7.0.3 Corollary (SWIC for Newton nondegenerate hypersurfaces)**.** *The Seiberg–Witten invariant conjecture holds for Newton nondegenerate hypersurface singularities (see subsection 2.7).* □

Proof of theorem 7.0.2. If the graph G contains a single node, that is, $|\mathcal{N}| = 1$, then eq. 7.1 follows from lemma 7.2.1. If $\bar{v}(i)$ is a central node (and not the only node), then eq. 7.1 follows from lemma 7.6.6. If G contains exactly one or two nondegenerate arms and $\bar{v}(i)$ is not central, then eq. 7.1 follows from lemma 7.7.1. If G contains three nondegenerate arms and $\bar{v}(i)$ is not central, then eq. 7.1 follows from lemma 7.8.2. By proposition 3.6.4, there are no other cases to consider.

Summing the left hand side of eq. 7.1 gives a telescopic series yielding $q_{Z_K} - q_0$. We have $q_0 = 0$ because $Z_0(t)$ is supported on the Lipman cone $\mathcal{S}_{\mathrm{top}} \subset \mathbb{Z}_{\geq 0}\langle\mathcal{V}\rangle$, and $q_{Z_K} = \mathbf{sw}^0_M(\sigma_{\mathrm{can}}) - (Z_K^2 + |\mathcal{V}|)/8$ by proposition 2.6.9. □

7.1 Coefficients of the reduced zeta function

In this subsection we will describe a reduction process which will simplify the proof, as well as computing the coefficients of the reduced zeta function. The reduction is a special case of a general reduction theory established by László [19].

7.1.1 Definition. Define $L^{\mathcal{N}} = \mathbb{Z}\langle E_n | n \in \mathcal{N}\rangle \subset L$ and let $\pi^{\mathcal{N}} : L \to L^{\mathcal{N}}$ be the canonical projection. Set $V'_Z = \mathbb{Z}\langle E^*_v | v \in \mathcal{N} \cup \mathcal{E}\rangle$ and $V_Z = V'_Z \cap L$ and $V_Z^{\mathcal{N}} = \pi^{\mathcal{N}}(V_Z)$. For $l \in L$ we also write $\pi^{\mathcal{N}}(l) = l|_{\mathcal{N}}$.

7.1.2 Lemma. *We have*

$$V_Z^{\mathcal{N}} = \left\{ l \in L^{\mathcal{N}} \,\middle|\, \forall n \in \mathcal{N},\, n' \in \mathcal{N}_n : \frac{\beta_{n,n'} m_n(l) + m_{n'}(l)}{\alpha_{n,n'}} \in \mathbb{Z} \right\}. \tag{7.3}$$

Furthermore, assuming $l' \in V_Z$ with $l'|_{\mathcal{N}} = l$ and $n \in \mathcal{N}$, then, for $n' \in \mathcal{N}_n$, we have

$$m_u(l') = \frac{\beta_{n,n'} m_n(l) + m_{n'}(l)}{\alpha_{n,n'}}, \tag{7.4}$$

where $u = u_{n,n'}$, and for $n' \in \mathcal{N}^*_n \setminus \mathcal{N}$

$$m_u(l') = \frac{\beta_{n,n'} m_n(l) - (l', E_e)}{\alpha_{n,n'}}, \tag{7.5}$$

where $e \in \mathcal{E}_n$ so that $n' = n^*_e$ and, again, $u = u_{n,n'}$.

Proof. We start by noting that eqs. (7.4) and (7.5) follow from remark 3.2.2(ii). In fact, this proves the inclusion $\subset$ in eq. 7.3. By further application of remark 3.2.2(ii), given an l in the right hand side of eq. 7.3, $n \in \mathcal{N}$ and $n' \in \mathcal{N}_n$, we can construct a sequence $m_{v_1}(l'), \dots, m_{v_s}(l')$ between $m_n(l') := m_n(l)$ and $m_{n'}(l') := m_{n'}(l)$, where $v_1, \dots, v_s$ are as in fig. 1. In fact, we have $m_{v_1}(l') = (\beta_{n,n'} m_n(l) + m_{n'}(l))/\alpha_{n,n'}$ and the other multiplicities are determined by $m_{v_{s-1}}(l') - b_{v_s} m_{v_s}(l') + m_{v_{s+1}}(l') = 0$. For $n' \in \mathcal{N}_n^* \setminus \mathcal{N}$, we can choose $m_u(l')$ randomly for $u = u_{n,n'}$ and construct a similar sequence. This yields an element $l' \in L$ satisfying $(l', E_v) = 0$ for any $v \in \mathcal{V}$ with $\delta_v = 2$, that is, $l' \in V_Z$, proving the inclusion $\supset$ in eq. 7.3, hence equality. $\square$

7.1.3 Definition. For any $e \in \mathcal{E}$, set $D_e = \alpha_e E_e^* - E_{n_e}^*$.

7.1.4 Lemma. *Let $n \in \mathcal{N}$ and $e \in \mathcal{E}_n$. Then D_e is an effective integral which is supported on the leg containing e. In fact, the family $(D_e)_{e\in\mathcal{E}}$ is a $\mathbb{Z}$-basis for* $\ker(V_Z \to V_Z^{\mathcal{N}})$.

Proof. First, if $v \in \mathcal{V}$ is a vertex outside the leg containing e, then $I_{n,v}^{-1} = \alpha_e I_{e,v}^{-1}$ (recall the notation for the intersection matrix and its inverse, definition 2.2.6). This follows from [51], Theorem 12.2, see also [9], Lemma 20.2. Thus, $m_v(D_e) = 0$ for $v \in \mathcal{V}$ not on the leg, i.e. D_e is supported on the leg. Let u_e be the neighbour of n on this leg. We find $m_{u_e}(D_e) = (E_n, D_e) = 1$. Furthermore, if the leg consists of vertices $v_1, \dots, v_s$ as in fig. 1, then the equations $m_{v_{r-1}}(D_e) - b_{v_r} m_{v_r}(D_e) + m_{v_{r+1}}(D_e) = 0$ recursively show that $m_{v_r} \in \mathbb{Z}$ for all r. Thus, we have $D_e \in L$. Since $(D_e, E_v) \leq 0$ for all v on the leg, we find, by lemma 2.2.12, that $m_v(D_e) > 0$ for any such v, that is, D_e is effective and its support is the leg.

For the last statement, set $K = \ker(V_Z \to V_Z^{\mathcal{N}})$. Note first that by 7.1.2 we have $\operatorname{rk} K = |\mathcal{E}|$. It is then enough to find a dual basis, that is, $\lambda_e \in \operatorname{Hom}(K, \mathbb{Z})$ satisfying $\lambda_e(D_{e'}) = \delta_{e,e'}$. By what we have just shown, this is satisfied by $\lambda_e(l) = m_{u_e}(l)$. $\square$

7.1.5 Definition. Recall the definition of the Lipman cone $\mathcal{S}_{\mathrm{top}}$ in definition 2.3.1. Set $\mathcal{S}_Z = \mathcal{S}_{\mathrm{top}} \cap V_Z$ and for $l \in V_Z^{\mathcal{N}}$, define $\mathcal{S}_Z(l) = \mathcal{S}_Z \cap (\pi^{\mathcal{N}})^{-1}(l)$. Define also $\mathcal{S}_Z^{\mathcal{N}} = \pi^{\mathcal{N}}(\mathcal{S}_Z)$.

7.1.6 Lemma. *Let $l \in V_Z^{\mathcal{N}}$ and choose $l' \in V_Z$ so that $l'|_{\mathcal{N}} = l$. The element*

$$\psi(l) = l' - \sum_{e\in\mathcal{E}} \left\lfloor \frac{(-l', E_e)}{\alpha_e} \right\rfloor D_e \tag{7.6}$$

is independent of the choice of l'. Furthermore, the set $\mathcal{S}_Z(l)$ consists of the elements $\psi(l) + \sum_{e\in\mathcal{E}} k_e D_e$ where $k_e \in \mathbb{N}$ satisfy $\sum_{e\in\mathcal{E}_n} k_e \leq (-\psi(l), E_n)$ for all $n \in \mathcal{N}$.

Proof. Let ψ' be the element on the right hand side of eq. 7.6. For any $l'' \in V_Z$, also satisfying $l''|_{\mathcal{N}} = l$ define ψ'' similarly, using l''. By lemma 7.1.4, there exist $k_e \in \mathbb{Z}$ for $e \in \mathcal{E}$, so that $l'' = \psi' + \sum_{e\in\mathcal{E}} k_e D_e$. By definition, we have $0 \leq (-\psi', E_e) < \alpha_e$, and so $k_e = \left\lfloor \frac{(-l'', E_e)}{\alpha_e} \right\rfloor$, which gives $\psi'' = \psi'$.

For the second statement, we note first that by lemma 7.1.4, any element $l' \in V_Z$, restricting to l, is of the form $\psi(l) + \sum_{e\in\mathcal{E}} k_e D_e$ for some $k_e \in \mathbb{Z}$.

Also, we have $l' \in \mathcal{S}_Z(l)$ if and only if $(l', E_v) \leq 0$ for all $v \in \mathcal{E} \cup \mathcal{N}$. For $e \in \mathcal{E}$ we have $-\alpha_e < (\psi(l), E_e) \leq 0$ and $(l', E_e) = (\psi(l), E_e) - k_e\alpha_e$, showing $(l', E_e) \leq 0$ if and only if $k_e \geq 0$. Using lemma 7.1.4 and the results found in its proof, we find $(l', E_n) = (\psi(l), E_n) + \sum_{e\in\mathcal{E}} k_e$. Thus, $(l', E_n) \leq 0$ if and only if $\sum_{e\in\mathcal{E}} k_e \leq (-\psi(l), E_n)$. □

7.1.7 Remark. Let $l \in V_Z^{\mathcal{N}}$. By lemma 7.1.6, we have $\mathcal{S}_Z(l) \neq \emptyset$ if and only if $\psi(l) \in \mathcal{S}_Z$, which is equivalent to $(\psi(l), E_n) \leq 0$ for all $n \in \mathcal{N}$.

7.1.8 Lemma. *Let $l \in V_Z$ and $e \in \mathcal{E}$. If $u = u_e$, then*

$$m_u(\psi(l)) = \left\lceil \frac{\beta_e m_n(l)}{\alpha_e} \right\rceil.$$

Proof. This follows from eq. 7.5 and the fact that $0 \leq (-\psi(l), E_e) < \alpha_e$. □

7.1.9 Lemma. *Let $l' \in \mathcal{S}_Z$ and take $Z \in \mathcal{S}_{\mathrm{top}}$ satisfying $Z = x(Z)$ (see subsection 5.1). Then $l' \geq Z$ if and only if $l'|_{\mathcal{N}} \geq Z|_{\mathcal{N}}$.*

Proof. The "only if" part of the statement is trivial. For the "if" part, take $l' \in \mathcal{S}_Z(l)$. We find $l' \geq x(l')$ by the definition of x. The result therefore follows from the monotonicity of x, proposition 5.1.4(i). □

7.1.10 Definition. Define the *reduced zeta function* $Z_0^{\mathcal{N}}(t)$ in $|\mathcal{N}|$ variables by setting $t_v = 1$ in $Z_0(t)$ if $v \notin \mathcal{N}$. Thus, we have $Z_0^{\mathcal{N}}(t) = \sum_{l\in L^{\mathcal{N}}} z_l^{\mathcal{N}} t^l$ where $z_l^{\mathcal{N}} = \sum\{z_{l'} \,|\, l' \in \mathcal{S}_Z(l)\}$. This series is supported on $S_Z^{\mathcal{N}}$.

7.1.11. Take $l' \in V_Z$ and write $l' = \sum_{v\in\mathcal{N}\cup\mathcal{E}} a_v E_v^*$. Using eq. 2.1 and the linear independence of the family $(E_v^*)_{v\in\mathcal{V}}$, we see that $z_{l'} = \prod_{v\in\mathcal{N}\cup\mathcal{E}} z_{l',v}$, where we set

$$z_{l',v} = \begin{cases} 1 & \text{if } v \in \mathcal{E},\ 0 \leq a_v, \\ (-1)^{a_v}\dbinom{\delta_v - 2}{a_v} & \text{if } v \in \mathcal{N},\ 0 \leq a_v \leq \delta_v - 2, \\ 0 & \text{otherwise.} \end{cases}$$

For any $l \in L^{\mathcal{N}}$, we therefore have, using lemma 7.1.6,

$$\begin{aligned} z_l^{\mathcal{N}} &= \sum_{\substack{(k_e)\in\mathbb{N}^{\mathcal{E}} \\ \forall n'\in\mathcal{N}:\, \sum_{e\in\mathcal{E}_{n'}} k_e \leq (-\psi(l), E_{n'})}} \prod_{n\in\mathcal{N}} z_{\psi(l)+\sum_e k_e D_e, n} \\ &= \prod_{n\in\mathcal{N}} \sum_{\substack{(k_e)\in\mathbb{N}^{\mathcal{E}_n} \\ \sum_{e\in\mathcal{E}_n} k_e \leq (-\psi(l), E_n)}} (-1)^{(-\psi(l),E_n) - \sum_{e\in\mathcal{E}_n} k_e} \binom{\delta_n - 2}{(-\psi(l), E_n) - \sum_{e\in\mathcal{E}_n} k_e}. \end{aligned} \tag{7.7}$$

Define $z_{l,n}^{\mathcal{N}}$ as the n^{th} factor in the product on the right hand side above, so that $z_l^{\mathcal{N}} = \prod_{n\in\mathcal{N}} z_{l,n}^{\mathcal{N}}$.

7.1.12 Lemma. *Let $l \in V_Z^{\mathcal{N}}$ and $n \in \mathcal{N}$.*

(i) If $\delta_n - |\mathcal{E}_n| = 1$, then

$$z_{l,n}^{\mathcal{N}} = \begin{cases} 1 & \text{if } (-\psi(l), E_n) \geq 0, \\ 0 & \text{else.} \end{cases}$$

(ii) If $\delta_n - |\mathcal{E}_n| = 2$, then

$$z^{\mathcal{N}}_{l,n} = \begin{cases} 1 & \text{if } (-\psi(l), E_n) = 0, \\ 0 & \text{else.} \end{cases}$$

(iii) If $\delta_n - |\mathcal{E}_n| = 3$, then

$$z^{\mathcal{N}}_{l,n} = \begin{cases} 1 & \text{if } (-\psi(l), E_n) = 0, \\ -1 & \text{if } (-\psi(l), E_n) = 1, \\ 0 & \text{else.} \end{cases}$$

(iv) If $\delta_n - |\mathcal{E}_n| = 0$, then $z^{\mathcal{N}}_{l,n} = \max\{0, (-\psi(l), E_n) + 1\}$.

Proof. From eq. 7.7, we find (setting $k = \sum_{e \in \mathcal{E}_n} k_e$)

$$z^{\mathcal{N}}_{l,n} = \sum_{k=0}^{(-\psi(l),E_n)} (-1)^{(-\psi(l),E_n)-k} \binom{|\mathcal{E}_n| + k - 1}{k} \binom{\delta_n - 2}{(-\psi(l), E_n) - k} = c_{(-\psi(l),E_n)}$$

where we set $C(t) = \sum_{k=0}^{\infty} c_k t^k = A(t) \cdot B(t)$, where

$$A(t) = \sum_{k=0}^{\infty} \binom{|\mathcal{E}_n| + k - 1}{k} t^k = (1-t)^{-|\mathcal{E}_n|},$$
$$B(t) = \sum_{k=0}^{\infty} (-1)^k \binom{\delta_n - 2}{k} t^k = (1-t)^{\delta_n - 2},$$

hence $C(t) = (1-t)^{\delta_n - 2 - |\mathcal{E}_n|}$. In each case, this proves the lemma. □

7.1.13 Lemma. *We have*

$$q_{\bar{Z}_{i+1}} - q_{\bar{Z}_i} = \sum \left\{ z^{\mathcal{N}}_l \,\middle|\, l \in V^{\mathcal{N}}_Z,\, l \geq \bar{Z}_i|_{\mathcal{N}},\, m_{\bar{v}(i)}(l) = m_{\bar{v}(i)}(\bar{Z}_i) \right\}.$$

Proof. By definition, $q_{\bar{Z}_{i+1}}$ is the sum of $z_{l'}$ for $l' \in V_Z$ with $l' \not\geq \bar{Z}_{i+1}$. Subtracting $q_{\bar{Z}_i}$, we cancel out those summands for which $l' \not\geq \bar{Z}_i$. Note that these all appear in the formula for $q_{\bar{Z}_{i+1}}$ since $\bar{Z}_{i+1} > \bar{Z}_i$. Thus, by lemma 7.1.9 and the definition of $z^{\mathcal{N}}_l$, we have $q_{\bar{Z}_{i+1}} - q_{\bar{Z}_i} = \sum \left\{ z^{\mathcal{N}}_l \,\middle|\, l \in V^{\mathcal{N}}_Z,\, l \geq \bar{Z}_i|_{\mathcal{N}},\, l \not\geq \bar{Z}_{i+1}|_{\mathcal{N}} \right\}$. Since $\bar{Z}_{i+1}|_{\mathcal{N}} = \bar{Z}_i|_{\mathcal{N}} + E_{\bar{v}(i)}$, the condition $l \not\geq \bar{Z}_{i+1}|_{\mathcal{N}}$ is equivalent to $m_{\bar{v}(i)}(l) = m_{\bar{v}(i)}(\bar{Z}_i)$, assuming $l \geq \bar{Z}_i|_{\mathcal{N}}$. □

7.1.14 Definition. For each step i in the computation sequence, set

$$S_i = \left\{ l \in V^{\mathcal{N}}_Z \,\middle|\, l \geq \bar{Z}_i|_{\mathcal{N}},\, m_{\bar{v}(i)}(l) = m_{\bar{v}(i)}(\bar{Z}_i),\, z^{\mathcal{N}}_l \neq 0 \right\}.$$

7.1.15 Corollary. *For each i, eq. 7.1 is equivalent to*

$$\sum_{l \in S_i} z^{\mathcal{N}}_l = |\bar{P}_i|.$$

□

7.2 The one node case

In the case when the diagram $\Gamma(f)$ contains only a single face, the graph G is starshaped, i.e. contains a single node n_0. Our function f then has the form $f = f_{n_0} + f^+$, where $\mathrm{wt}_{n_0}(f^+) > \mathrm{wt}_{n_0}(f)$. The deformation $f_t(x) = f_{n_0} + tf^+$ has constant topological type, so for computations involving the zeta function, or any other topological invariant, we may assume that $f = f_{n_0}$, i.e. that f is weighted homogeneous. In other words, the variety $X = \{f = 0\} \subset \mathbb{C}^3$ has a good $\mathbb{C}^*$ action. The singularities of such varieties have been studied in [59, 49, 41] (to name a few).

7.2.1 Lemma. *Assume that $\mathcal{N} = \{n_0\}$. For any i we have $\bar{v}(i) = n_0$ and there is at most one element $l_i \in S_i$. In that case, we have*

$$z_{l_i}^{\mathcal{N}} = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}. \tag{7.8}$$

In particular, eq. 7.1 holds.

Proof. It is clear that $\bar{v}(i) = n_0$ for all i and that $m_{n_0}(l_i) = i$ determines a unique element $l_i \in L^{\mathcal{N}} = V_{\mathbb{Z}}^{\mathcal{N}} \cong \mathbb{Z}$ (for $L^{\mathcal{N}} = V_Z^{\mathcal{N}}$, see lemma 7.1.2). By lemma 7.1.12(iv), we have $z_{l_i}^{\mathcal{N}} = \max\{0, (-\psi(l_i), E_{\bar{v}(i)}) + 1\}$. By lemmas 5.1.5 and 7.1.8 we have $m_u(\bar{Z}_i) = m_u(\psi(l_i))$ for $u \in \mathcal{V}_{\bar{v}(i)}$ and, furthermore, $m_{\bar{v}(i)}(\bar{Z}_i) = i = m_{\bar{v}(i)}(\psi(l_i))$. Therefore, $(-\psi(l_i), E_{\bar{v}(i)}) = (-\bar{Z}_i, E_{\bar{v}(i)})$, proving eq. 7.8. □

7.3 Multiplicities along arms

In this subsection we use lemma 7.1.12 to determine multiplicities along arms given "local data", i.e. multiplicities on two nodes. Recall the definition of arms in subsection 3.6.

7.3.1. Assume that the diagram $\Gamma(f)$ has a nondegenerate arm consisting of faces $F_{n_1}, \ldots, F_{n_j}$ so that for $s = 2, \ldots, j-1$ we have $\mathcal{N}_{n_s} = \{n_{s-1}, n_{s+1}\}$ and $\mathcal{N}_{n_j} = \{n_{j-1}\}$. In this case, we either have $\mathcal{N}_{n_1} = \{n_2\}$, or there is a node $n_0 \in \mathcal{N}$ so that $\mathcal{N}_{n_1} = \{n_0, n_2\}$ or $\{n_0\}$, depending on whether $j > 1$ or $j = 1$. If there is such a node n_0, then we set $\nu = 0$, otherwise, set $\nu = 1$. Note that if $\nu = 1$, then $\mathcal{N} = \{n_1, \ldots, n_j\}$.

We fix the following notation as well. Let $\alpha_s = \alpha_{n_s, n_{s+1}}$ and $\beta_s = \beta_{n_s, n_{s+1}}$, for $\nu \leq s < j$. Also, let $\overline{\beta}_s = \beta_{n_{s+1}, n_s}$, so that $\beta_s \overline{\beta}_s \equiv 1 \,(\mathrm{mod}\, \alpha_s)$. This way, the two equations $\beta_s m_s + m_{s+1} \equiv 0$ and $m_s + \overline{\beta}_s m_{s+1} \equiv 0 \,(\mathrm{mod}\, \alpha_s)$ are equivalent.

We always assume that $\nu < j$. If $\nu = j$, then we necessarily have $\nu = j = 1$ and $\mathcal{N} = \{n_1\}$. This case is covered in subsection 7.2. Note that lemma 7.3.8 does not make sense unless we make this assumption.

7.3.2 Lemma. *Assume the notation given in 7.3.1. Let $\nu \leq s < j$ and assume that we have numbers m_s, m_{s+1} satisfying $\beta_s m_s + m_{s+1} \equiv 0 \,(\mathrm{mod}\, \alpha_s)$ Then there exist unique numbers $m_\nu, \ldots, m_j$ (with m_s and m_{s+1} unchanged), so that for any r we have*

$$\beta_r m_r + m_{r+1} \equiv 0 \,(\mathrm{mod}\, \alpha_r) \tag{7.9}$$

and

$$\frac{m_{r-1} + \bar{\beta}_{r-1} m_r}{\alpha_{r-1}} + E_{n_r}^2 m_r + \frac{\beta_r m_r + m_{r+1}}{\alpha_{r+1}} + \sum_{e \in \mathcal{E}_{n_r}} \left\lceil \frac{\beta_e m_r}{\alpha_e} \right\rceil = 0. \tag{7.10}$$

7.3.3 Remark. Assume that $\nu < r < j$. If $l \in L^{\mathcal{N}}$ and $m_{n_s}(l) = m_s$ for $s = r-1, r, r+1$, then 7.10 is equivalent to $z^{\mathcal{N}}_{l,n_r} \neq 0$, which again is equivalent to $z^{\mathcal{N}}_{l,n_r} = 1$. This follow from lemma 7.1.12(ii), and the fact that the left hand side of eq. 7.10 equals $(\psi(l), E_{n_r})$ by lemmas 7.1.2 and 7.1.8.

Proof of lemma 7.3.2. Assume that $\nu \leq r < j$ and that we have integers m_r and m_{r+1} satisfying $\beta_r m_r + m_{r+1} \equiv 0 \,(\mathrm{mod}\,\alpha_r)$. Then eq. 7.10 defines an integer $m_{n_{r-1}}$ which satisfies eq. 7.10. It is clear from this definition that $m_{r-1} + \overline{\beta}_{r-1} m_r \equiv 0 \,(\mathrm{mod}\,\alpha_{r-1})$, or equivalently, $\beta_{r-1} m_{r-1} + m_r \equiv 0 \,(\mathrm{mod}\,\alpha_{r-1})$. This way, we obtain $m_\nu, \ldots, m_{s-1}$ recursively. A similar process produces the numbers $m_{s+2}, \ldots m_j$. $\square$

7.3.4 Definition. We will refer to a sequence of numbers $m_\nu, \ldots, m_j \in \mathbb{Z}$ satisfying eqs. (7.9) and (7.10) as an *arm sequence*. When there are more than one arms in the diagram, it will be clear from context which arm is being referred to.

7.3.5 Remark. We have $\delta_{n_r} - |\mathcal{E}_{n_r}| = |\mathcal{N}_{n_r}| = 2$ for $\nu < r < j$ (for $1 < r < j$ if there is no n_0). Therefore, it follows form lemma 7.1.12 and lemma 7.1.8 that if $l \in L^{\mathcal{N}}$ and $z^{\mathcal{N}}_l \neq 0$ then the sequence given by $m_r = m_{n_r}(l)$ must be an arm sequence.

7.3.6 Lemma. *Let $m_\nu, \ldots, m_j$ be an arm sequence. There exist unique points $p_{\nu+1}, \ldots, p_{j-1} \in \mathbb{Z}^3$ so that for each $\nu < s < j$ and $r = s-1, s, s+1$ we have $\ell_{n_r}(p_s) = m_r$.*

Proof. Let s be given, $\nu < s < j$. For simplicity, set $\ell_r = \ell_{n_r}$ for all r. We note first that the functionals $\ell_{s-1}, \ell_s, \ell_{s+1}$ are linearly independent. This follows from the fact that the functions $\ell_r - \mathrm{wt}_{n_r}(f)$ for $r = s-1, s+1$ restricted to the hyperplane $\ell_s = \mathrm{wt}_{n_s}(f)$ support adjacent edges of the polygon F_{n_s}. Thus, ℓ_{s-1}, ℓ_{s+1} induce an isomorphism $H^{=}(\mathrm{wt}(f)) \to \mathbb{R}^2$, so the three functions form a dual basis of $\mathbb{R}^3$. The existence of $p_s \in \mathbb{R}^3$ follows, but we must show that p_s has integral coordinates.

Define $u_+, u_-, u_0 \in \mathcal{V}_{n_s}$ by $u_\pm = u_{n_s, n_{s\pm1}}$, and let u_0 be some other neighbour. Since the functional ℓ_{n_s} is primitive, the hyperplane $H = H^{=}_{n_s}(m_s)$ contains a two dimensional affine lattice $H \cap \mathbb{Z}^3$. The restrictions $\ell_{u_\pm}|_H, \ell_{u_0}|_H$ are all primitive, and by corollary 4.1.7 the functions $\ell_{u_+}|_H, \ell_{u_0}|_H$ give affine coordinates over $\mathbb{Z}$ of this lattice. It is therefore enough to show that these functionals take integral values on p_s. First, we find

$$\ell_{u_+}(p_s) = \left(\frac{\beta_s \ell_{n_s} + \ell_{n_{s+1}}}{\alpha_s}\right)(p_s) = \frac{\beta_s m_s + m_{s+1}}{\alpha_s} \in \mathbb{Z}$$

by eq. 7.9, and a similar formula for $\ell_{u_-}(p_s)$. Subtracting eq. 3.1 from eq. 7.10, evaluated at p_s, and dividing by $|\mathcal{E}_n|$ one finds

$$\ell_{u_0}(p_s) = \left\lceil \frac{\beta_e m_s}{\alpha_e} \right\rceil \in \mathbb{Z}, \tag{7.11}$$

where $e \in \mathcal{E}_{n_s}$. Note that here we use the fact that ℓ_{u_0} does not depend on the choice of $u_0 \in \mathcal{V}_{n_s}$, as long as $u_0 \neq u_\pm$, and similarly, α_e, β_e do not depend on $e \in \mathcal{E}_n$. This is because F_{n_s} is a triangle, and all legs of n_s are associated with one edge of this triangle, see also proposition 3.6.5. $\square$

7.3.7 Definition. Let $m_\nu, \dots, m_j$ be an arm sequence. We call the points $p_{\nu+1}, \dots, p_{j-1}$ the *associated vertices.* The *associated lines* are defined as $L_s = \{p \in \mathbb{R}^3 \,|\, \ell_{n_s}(p) = m_s,\, \ell_{n_{s+1}}(p) = m_{s+1}\}$ for $\nu \leq s < j$. Thus, we have $p_s, p_{s+1} \in L_s$, whenever these are defined.

7.3.8 Lemma. *Let $m_\nu, \dots, m_j$ be an arm sequence, and assume that the arm goes in the direction of the x_3-axis. Assume furthermore that $l \in V_Z^{\mathcal{N}}$ with $m_{n_s}(l) = m_s$ for all s. The following are equivalent.*

(i) There is an $\nu \leq s < j$ so that L_s contains an integral point with nonnegative x_1 and x_2 coordinates.

(ii) The line L_s contains an integral point with nonnegative x_1 and x_2 coordinates for all $0 \leq s < j$.

(iii) We have

$$\frac{m_{j-1} + \bar{\beta}_{j-1} m_{j-1}}{\alpha_{j-1}} - b_{n_j} m_j + \sum_{e \in \mathcal{E}_{n_j}} \left\lceil \frac{\beta_e m_j}{\alpha_e} \right\rceil \leq 0. \tag{7.12}$$

(iv) We have $z_{l,n_j}^{\mathcal{N}} \neq 0$.

(v) We have $z_{l,n_j}^{\mathcal{N}} = 1$.

Proof. Using lemmas 7.1.8 and 7.1.12, we see that (iii), (iv) and (v) are equivalent. For brevity, let us say (in this proof) that $p \in \mathbb{R}^3$ is *good* if it has integral coordinates, with the x_1 and x_2 coordinates nonnegative. Let $p_{\nu+1}, \dots, p_{j-1}$ be the points associated with the arm sequence. We start by proving the following

Claim. Assume that L_s contains a good point for some $\nu \leq s < j$. Then p_s is a good point if $s > \nu$, and p_{s+1} is good if $s + 1 < j$.

We prove the claim for p_s, the proof for p_{s+1} is the same. By proposition 3.6.5, F_{n_s} is a triangle with exactly one edge on the boundary $\partial\Gamma(f)$, and we can assume that this edge lies on the $x_2 x_3$ plane. For $k = 1, 2, 3$, let ℓ_k be the standard coordinate functions in $\mathbb{R}^3$, that is, $\ell_1(p) = \langle p, (1,0,0) \rangle$, etc. Using notation as in eq. 7.11, we find

$$\left(\frac{\beta_e \ell_{n_s} + \ell_1}{\alpha_e} \right)(p_s) = \ell_{u_0}(p_s) = \left\lceil \frac{\beta_e m_j}{\alpha_e} \right\rceil,$$

where $e \in \mathcal{E}_n$, which shows that $0 \leq \ell_1(p_s) < \alpha_e$. From proposition 3.2.10 we see that the restricted function $\ell_1|_{L_s}$ has content α_e. This shows that $\ell_1|_{L_s \cap \mathbb{Z}^3}$ takes its minimal nonnegative value at p_s. Since L_s is parallel to the edge $F_{n_s} \cap F_{n_{s+1}}$, we find that ℓ_1 and ℓ_2 define opposite orientations on L_s. Thus, if $\ell_2(p_s) < 0$, we have $\ell_2(p) < 0$ for all integral points $p \in L_s$ for which $\ell_1(p) \geq 0$. By the assumption, this is not the case, so $\ell_2(p_s) \geq 0$, proving the claim.

The implication (i)$\Rightarrow$(ii) now follows from repeated usage of the claim. Namely, if (i) holds for some $s < j - 1$, then $p_{s+1} \in L_s$ is good. But this means that $p_{s+1} \in L_{s+1}$ is good, and we can apply the claim to L_{s+1}. This proves that L_r contains good points for any $r \geq s$. A similar induction proves the same statement for $r < s$. Thus, (ii) holds.

Next, we prove (ii)$\Rightarrow$(iii). Let $p \in L_{j-1}$ be good. Subtracting eq. 3.1, with $n = n_j$ and evaluated at p, from eq. 7.12, we see that it is enough to prove $\lceil \beta_e m_j/\alpha_e \rceil \leq \ell_u(p)$ for any $e \in \mathcal{E}_{n_j}$, with $u = u_e$. Let $n' = n_e^*$. By proposition 3.6.5, $\ell_{n'}$ is a nonnegative linear combination of ℓ_1 and ℓ_2. Therefore, we have $\ell_{n'}(p) \geq 0$. The formula $\alpha_e \ell_u = \beta_e \ell_n + \ell_{n'}$ therefore gives $\ell_u(p) \geq \beta_e m_j/\alpha_e$, hence $\ell_u(p) \geq \lceil \beta_e m_j/\alpha_e \rceil$, since $\ell_u(p) \in \mathbb{Z}$.

Finally, we prove (iii)$\Rightarrow$(i). Assuming (iii), we will prove (i) with $s = j-1$. Let $u_- = u_{n_j,n_{j-1}}$ and let $\tilde\ell_1, \tilde\ell_2$ and $\mathcal{E}^1_{n_j}, \mathcal{E}^2_{n_j}$ be as in proposition 3.6.5. Take $e_1 \in \mathcal{E}^1_{n_j}$. Then $\tilde\ell_1 = \ell_1$ restricted to the line L_s has content α_{e_1} by proposition 3.2.10. Thus, there is a unique integral point $p \in L_s$ so that $0 \leq \ell_1(p) < \alpha_{e_1}$ and it suffices to show $\ell_2(p) \geq 0$. Subtract eq. 3.1 for $n = n_j$, evaluated at p from eq. 7.12 to find

$$\sum_{e \in \mathcal{E}_{n_j}} \left\lceil \frac{\beta_e m_j}{\alpha_e} \right\rceil - \ell_{u_e}(p) \leq 0. \tag{7.13}$$

We have

$$\ell_{u_{e_1}}(p) = \frac{\beta_e m_j + \ell_1(p)}{\alpha_e} = \left\lceil \frac{\beta_e m_j}{\alpha_{e_1}} \right\rceil$$

by the definition of p, and the fact that $\ell_{u_{e_1}}(p) \in \mathbb{Z}$. Therefore, the summands in eq. 7.13 corresponding to $e \in \mathcal{E}^1_{n_j}$ vanish, and we are left with summands corresponding to $e \in \mathcal{E}^2_{n_j}$, yielding

$$\frac{\beta_e m_j + \tilde\ell_2(p)}{\alpha_e} = \ell_u(p) \geq \left\lceil \frac{\beta_e m_j}{\alpha_e} \right\rceil$$

for $e \in \mathcal{E}^2_{n_j}$, hence $\tilde\ell_2(p) \geq 0$. If $\tilde\ell_2 = \ell_2$, then we are done. Otherwise, we have $\tilde\ell_2 = \alpha_{e_1}\ell_2 + \ell_1$ so we find $\ell_2(p) \geq 0$, since $\ell_1(p) < \alpha_{e_1}$. □

7.3.9 Lemma. *Let $m_\nu, \ldots, m_j$ be a nonzero arm sequence and assume that $\nu < r < j$. Assume furthermore that the equivalent properties in 7.3.8 hold. Then, for $\nu < s < j$ we have*

$$\frac{m_{s-1}}{m_{s-1}(Z_K - E)} \leq \frac{m_s}{m_s(Z_K - E)} \Rightarrow \frac{m_s}{m_s(Z_K - E)} < \frac{m_{s+1}}{m_{s+1}(Z_K - E)} \tag{7.14}$$

and

$$\frac{m_{s+1}}{m_{s+1}(Z_K - E)} \leq \frac{m_s}{m_s(Z_K - E)} \Rightarrow \frac{m_s}{m_s(Z_K - E)} < \frac{m_{s-1}}{m_{s-1}(Z_K - E)}. \tag{7.15}$$

Proof. We will prove eq. 7.14, eq. 7.15 follows similarly. Assume that the arm goes in the direction of the x_3 coordinate and let $p_{\nu+1}, \ldots, p_{j-1}$ be the associated vertices. Let $B_r = \cup_{t=r}^j C_{n_r}(Z_K - E)$. The functional

$$\ell_r = \frac{\ell_{n_{r-1}}}{m_{r-1}(Z_K - E)} - \frac{\ell_{n_r}}{m_r(Z_K - E)}$$

seperates the diagram $\Gamma(Z_K - E)$ into two parts, namely $F_{n_r}(Z_K - E), \ldots, F_{n_j}(Z_K - E)$, where it is nonnegative, and the other faces where it is nonpositive. Therefore, $p_s \in B_r$ if and only if $\ell_r(p_s) \geq 0$, with equality if and ony if $p_s \in \partial B_r$. Thus, the left hand side of eq. 7.14 gives $p_s \notin B_s^\circ$, which gives $p_s \notin B_{s+1}$, which, again, translates to the right hand side of eq. 7.14. □

7.4 Multiplicities around $v(i)$

In this subsection we assume a fixed step i of the computation sequence from definition 5.2.2. We also assume $|\mathcal{N}| > 1$.

7.4.1 Lemma. *Let $u \in \mathcal{V}_{\bar{v}(i)}$ and assume $\bar{P}_i \neq \emptyset$. Then*

$$m_u(\bar{Z}_i) = \min\left\{\ell_u(p) \,\middle|\, p \in \bar{P}_i\right\}.$$

Proof. By lemma 5.3.10 we have

$$\bar{P}_i = \left\{p \in H^{=}_{\bar{v}(i)}(\bar{Z}_i) \cap \mathbb{Z}^3 \,\middle|\, \forall u \in \mathcal{V}_{\bar{v}(i)} : \ell_u(p) \geq m_u(\bar{Z}_i)\right\}.$$

It is therefore enough to show that for any $u \in \mathcal{V}_{\bar{v}(i)}$, there is a $p \in \bar{P}_i$ so that $\ell_u(p) = m_u(\bar{Z}_i)$. By corollary 4.1.7, there is a $u' \in \mathcal{V}_{\bar{v}(i)}$ so that $\ell_u, \ell_{u'}$ form an affine basis when restricted to $H^{=}_{\bar{v}(i)}(\bar{Z}_i)$. Therefore, there is a $p \in H^{=}_{\bar{v}(i)}(\bar{Z}_i)$ so that $\ell_u(p) = m_u(\bar{Z}_i)$ and $\ell_{u'}(p) = m_{u'}(\bar{Z}_i)$. If $F_{v(i)}$ is a triangle, then there is a $u'' \in \mathcal{V}_{\bar{v}(i)}$ so that u, u', u'' represent all bamboos and leg groups of $\bar{v}(i)$. Furthermore, we must have $\ell_{u''}(p) \geq m_{u''}(\bar{Z}_i)$, since otherwise, by the above description, we would have $\bar{P}_i = \emptyset$, a contradiction.

Assume now that $F_{\bar{v}(i)}$ is a trapezoid. If u lies on a bamboo not corresponding to the top edge of $F_{\bar{v}(i)}(f)$ (see definition 4.1.6), then we may choose u' with the same property. Now define p in the same way as above (note that all vertices of a trapezoid are regular). It is then easy to see (from e.g. proposition 4.1.5) that for any $u'' \in \mathcal{V}_{\bar{v}(i)}$ with $\ell_{u''} \neq \ell_u, \ell_{u'}$, the function $\ell_{u''}$ restricted to the cone

$$\left\{p' \in H^{=}_{\bar{v}(i)}(\bar{Z}_i) \,\middle|\, \ell_u(p') \geq m_u(\bar{Z}_i),\, \ell_{u'}(p') \geq m_{u'}(\bar{Z}_i)\right\}$$

takes its maximal value at the vertex p. From the assumption $\bar{P}_i \neq \emptyset$, we now find $\ell_{u''}(p) \geq m_{u''}(\bar{Z}_i)$ for all $u'' \in \mathcal{V}_{\bar{v}(i)}$ and therefore $p \in \bar{P}_i$.

The last case we must consider is when $F_{\bar{v}(i)}$ is a trapezoid and u lies on a bamboo corresponding to a top face. We have $\bar{P}_i \subset \bar{r}_i F_{\bar{v}(i)}(Z_K - E)$. Since the length of the interval $\ell_u(F_{\bar{v}(i)}(Z_K - E))$ is one, we find that if ℓ_u takes an integral value on $\bar{r}_i F_{\bar{v}(i)}(Z_K - E)$, then it must be $\lceil \bar{r}_i m_u(Z_K - E)\rceil$. In other words, if $p \in P_i$, then $\ell_u(p) = \lceil \bar{r}_i m_u(Z_K - E)\rceil$ (in the case $\bar{r}_i = 1$, this gives $\ell_u(p) = m_u(Z_K - E)$ or $\ell_u(p) = m_u(Z_K - E) + 1$, but in the latter case, the point p has a negative coordinate). This finishes the proof of the lemma. □

7.4.2 Corollary. *Assume that F_{n_0} is a trapezoid, and that $n_1 \in \mathcal{N}$ so that $F_{n_0} \cap F_{n_1}$ is the top edge of the trapezoid and that $\bar{v}(i) = n_0$. Then $\ell_u(p) = m_u(\bar{Z}_i)$ for all $p \in \bar{P}_i$, where $u = u_{n_0,n_1}$.*

Proof. This follows from the above proof. □

7.4.3 Lemma. *Assume the notation in 7.3.1 and that $\bar{v}(i) = n_r$ for some $\nu < r < j$. For any $l \in S_i$, there is a unique $p \in \bar{P}_i$ so that $m_n(l) = \ell_n(p)$ for all $n \in \mathcal{N}_{\bar{v}(i)}$.*

Proof. If $e \in \mathcal{E}_{\bar{v}(i)}$, then lemmas 5.1.5 and 7.1.8

$$m_{u_e}(\psi(l)) = \left\lceil \frac{\beta_e m_{\bar{v}(i)}(l)}{\alpha_e} \right\rceil = m_{u_e}(\bar{Z}_i).$$

Let $u_+, u_- \in \mathcal{V}_{\bar{v}(i)}$, $u_\pm = u_{\bar{v}(i),n_{r\pm 1}}$ and take $e \in \mathcal{E}_{\bar{v}(i)}$. By corollary 4.1.7, the functionals $\ell_{\bar{v}(i)}, \ell_{u_e}, \ell_{u_+}$ form a dual basis of $\mathbb{Z}^3$. Therefore, there is a $p \in \mathbb{Z}^3$ satisfying

$$\begin{aligned} \ell_{\bar{v}(i)}(p) &= m_{\bar{v}(i)}(l) = m_{\bar{v}(i)}(\bar{Z}_i), \\ \ell_{u_e}(p) &= m_{u_e}(\psi(l)) = m_{u_e}(\bar{Z}_i), \\ \ell_{u_+}(p) &= m_{u_+}(\psi(l)) = \frac{\beta_r m_{\bar{v}(i)}(l) + m_{n_{r+1}}(l)}{\alpha_r} \geq m_{u_+}(\bar{Z}_i). \end{aligned} \tag{7.16}$$

The formula for $\ell_{u_+}(p)$ gives an integer by lemma 7.1.2. Furthermore, the inequality holds by lemma 5.1.5, using $l \geq \bar{Z}_i|_{\mathcal{N}}$. We have therefore shown that $\ell_u(p) = m_u(\psi(l))$ for all $u \in \mathcal{V}_{\bar{v}(i)}$ except for u_-. But, since $z_l^{\mathcal{N}} \neq 0$, we have $(\psi(l), E_{\bar{v}(i)}) = 0$ by lemma 7.1.12, hence

$$-b_{\bar{v}(i)} m_{\bar{v}(i)}(\psi(l)) + \sum_{u \in \mathcal{V}_{\bar{v}(i)}} m_u(\psi(l)) = 0 = -b_{\bar{v}(i)} \ell_{\bar{v}(i)}(p) + \sum_{u \in \mathcal{V}_{\bar{v}(i)}} \ell_u(p).$$

Cancelling out, we obtain $m_{u_-}(\psi(l)) = \ell_{u_-}(p)$ as well. This shows that we could have replaced the third equation in eq. 7.16 with a corresponding line with u_+ replaced by u_-. In particular, we have $\ell_{u_-}(p) = m_{u_-}(\psi(l)) \geq m_{u_-}(\bar{Z}_i)$. We have therefore shown $\ell_u(p) = m_u(\psi(l)) \geq m_u(\bar{Z}_i)$ for all $u \in \mathcal{V}_{\bar{v}(i)}$. By lemma 5.3.10 we have $p \in \bar{P}_i$. Now, we have $m_{n_{r+1}}(l) = \alpha_r m_{u_+}(\psi(l)) - m_{\bar{v}(i)}(l) = \alpha_r \ell_{u_+}(p) - \ell_{\bar{v}(i)}(p) = \ell_{n_{r+1}}(p)$, and $m_{n_{r-1}}(l) = \ell_{n_{r-1}}(p)$ similarly. Since the functionals ℓ_{n_s} with $s = r-1, r, r+1$ form a dual basis of $\mathbb{Q}^3$, uniqueness follows. □

7.4.4 Lemma. *Assume the notation in 7.3.1 and that either $\bar{v}(i) = n_j$, or $\nu = 1$ and $\bar{v}(i) = n_1$. For any $l \in S_i$ there is a unique $p \in \bar{P}_i$ so that $m_{n_{j-1}}(l) = \ell_{n_{j-1}}(p)$.*

Proof. We prove the lemma in the case when $\bar{v}(i) = n_j$, the case $\bar{v}(i) = n_1$ is similar.

Let $u_- = u_{n_j,n_{j-1}} \in \mathcal{V}_{\bar{v}(i)}$ as above and $u_0 = u_e \in \mathcal{V}_{\bar{v}(i)}$ for some $e \in \mathcal{E}_{n_j}$. If $\bar{v}(i)$ has a leg group with more than one element, choose u_0 from this leg group, otherwise choose u_0 arbitrarily. Then there is a unique $u_+ \in \mathcal{V}_{\bar{v}(i)}$ lying on a leg not in the same leg group as the leg containing u_0. Define $p \in \mathbb{Z}^3$ using eq. 7.16, but with u_+ and n_{r+1} replaced with u_- and n_{r-1} in the third line. Similarly as above, we find $p \in \mathbb{Z}^3$, as well as $\ell_{u_+}(p) \geq m_{u_+}(\psi(l)) = m_{u_+}(\bar{Z}_i)$ showing $p \in P_i$. The equation $m_{n_{r-1}} = \ell_{n_{r-1}}(p)$ now follows from $m_{u_-}(\psi(l)) = \ell_{u_-}(p)$ as above.

Next we prove uniqueness. Use the notation in proposition 3.6.5. We can assume that ℓ_v for $v = u_{n_j,n_{j-1}}, \bar{v}(i), u_{e_1}$ with $e_1 \in \mathcal{E}^1_{n_j}$ form a dual basis of $\mathbb{Z}^3$. Let $L_0, \ldots, L_{j-1}$ be the associated lines. We have $p \in L_{j-1} \cap \mathrm{conv}(F_{n_j}(Z_K - E) \cup \{0\}) \cap \mathbb{R}^3_{\geq 0}$. By proposition 3.2.10, we see that $\max_{F_{n_j}(Z_K-E)} \ell_1 = \max_{F_{n_j}} \ell_1 - 1 = \alpha_e - 1$. By the same lemma, the restriction $\ell_1|_{L_{j-1}}$ has content α_e. Therefore, p is determined as the unique point on L_{j-1} for which $0 \leq \ell_1(p) < \alpha_e$. □

7.4.5 Lemma. *Assume the same notation as above and assume that $v(i) = n_r$ for some $\nu < r \leq j$. Let $p \in \bar{P}_i$ be as defined in lemma 7.4.3 or lemma 7.4.4, depending on whether $r < j$ or $r = j$. Define $m_{r-1} = \ell_{n_{r-1}}(p)$ and $m_r = \ell_{n_r}(p)$. We have $\beta_{r-1} m_{r-1} + m_r \equiv 0 \,(\mathrm{mod}\, \alpha_{r-1})$, and we define an arm sequence $m_0, \ldots, m_j$ as in lemma 7.3.2, with associated vertices $p_{\nu+1}, \ldots, p_{j-1}$. Then $p_s = p$ for all $s \leq r$.*

Proof. By definition, it is equivalent to show $m_s = \ell_{n_s}(p)$ for $s \leq r$, as well as $m_{r+1} = \ell_{n_{r+1}}(p)$ in case $r < j$.

First, assume that $r < j$. Take $u_\pm = u_{n_r, n_{r\pm 1}} \in \mathcal{V}_{n_r}$. We have

$$\ell_{u_-}(p) = \frac{\ell_{n_{r-1}}(p) + \overline{\beta}_{r-1}\ell_{n_r}(p)}{\alpha_{r-1}} = \frac{m_{r-1} + \overline{\beta}_{r-1} m_r}{\alpha_{r-1}}.$$

Similarly as in the proof of lemma 7.4.4, we have $0 \leq \ell_{n_e^*}(p) < \alpha_e$, and so

$$\ell_u(p) = \frac{\beta_e m_r + \ell_{n_e^*}(p)}{\alpha_e} = \left\lceil \frac{\beta_e m_r}{\alpha_e} \right\rceil \tag{7.17}$$

for $u = u_e \in \mathcal{V}_{n_r}$, where $e \in \mathcal{E}_{n_r}$. Hence, subtracting eq. 7.10 from eq. 3.1 we get

$$\frac{\overline{\beta}_r \ell_{n_r}(p) + \ell_{n_{r+1}}(p)}{\alpha_r} = \frac{\beta_r m_r + m_{r+1}}{\alpha_r}$$

showing $\ell_{n_{r+1}}(p) = m_{r+1}$. This shows $p = p_r$. Next, we prove by descending induction that $p_s = p$ for $s < r$. Indeed, assuming that $p_{s+1} = p$, we have $\ell_{n_s}(p) = m_s$ and $\ell_{n_{s+1}} = m_{s+1}$. We can then follow the same procedure as above, once we prove eq. 7.17 for $u = u_e \in \mathcal{V}_{n_s}$ with $e \in \mathcal{E}_{n_s}$. Since $\alpha_e \ell_u = \beta_e \ell_{n_s} + \ell_{n_e^*}$, it is enough to prove $0 \leq \ell_{n_s}(p) < \alpha_e$. The first inequality is clear, since $p \in \mathbb{Z}^3_{\geq 0}$. For the second, by permutation of coordinates, we may assume that the arm $n_1, \ldots, n_j$ goes in the direction of the coordinate x_3, and that $\ell_{n_e^*} = \ell_1$. By construction, the projection of the sets $\mathrm{conv}(F_{n_t}) \cup \{(0,0,0)\}$ to the x_1x_2 plane lie within the triangle with vertices $(0,0)$, $(\alpha_e, 0)$ and $(0, a)$ for some $a \in \mathbb{Z}_{>0}$, for $t \geq s$, by proposition 3.2.10. In particular, we find, $\ell_{n_s}(p + (1,1,1)) \leq \min_{p' \in F_{n_r}} \ell_{n_s}(p' + (1,1,1)) \leq \alpha_r + \ell_{n_s}(1,1,1)$. Equality can only hold if $p + (1,1,1) = (\alpha_r, 0, *)$, which is impossible since $p \in \mathbb{Z}^3_{\geq 0}$. $\square$

7.5 Plan of the proof

The proof of theorem 7.0.2 will be broken into cases in the remaining subsections of this section, each dealing with various technical issues that arise. In this subsection we describe some general strategies common to these cases.

7.5.1. For any i and $p \in \bar{P}_i$, let $S_{i,p} = \left\{ l \in S_i \,\middle|\, \forall n \in \mathcal{N}_{\bar{v}(i)} : m_n(l) = \ell_n(p) \right\}$. Also, let $S_i' = S_i \setminus \cup_{p \in \bar{P}_i} S_{i,p}$. By lemmas 7.4.3 and 7.4.4, we have $S_i = \amalg_{p \in \bar{P}_i} S_{i,p}$ if $\bar{v}(i)$ is not a central vertex. By theorem 5.3.2, the right hand side of eq. 7.1 equals $|\bar{P}_i|$, while the left hand side is $\sum_{l \in S_i} z_l^{\mathcal{N}}$. Theorem 7.0.2 is therefore proved as soon as we prove the equations

$$\sum_{l \in S_i'} z_l^{\mathcal{N}} = 0 \tag{7.18}$$

and

$$\sum_{l \in S_{i,p}} z_l^{\mathcal{N}} = 1. \tag{7.19}$$

Although this is not always the case, we will follow this course of action in many of the cases.

7.5.2 Lemma. *Let $m_\nu, \ldots, m_j$ be an arm sequence as in definition 7.3.4 and assume that for some $r < j$ we have $m_r \geq m_r(\bar{Z}_i)$ and $m_{r+1} \geq m_{r+1}(\bar{Z}_i)$, as well as*

$$\frac{m_r}{m_r(Z_K - E)} \leq \frac{m_{r+1}}{m_{r+1}(Z_K - E)}. \tag{7.20}$$

Then $m_s \geq m_s(\bar{Z}_i)$ for all $s \geq r$. Similarly, if $r > \nu$ and $m_r \geq m_r(\bar{Z}_i)$ and $m_{r-1} \geq m_{r-1}(\bar{Z}_i)$, as well as

$$\frac{m_r}{m_r(Z_K - E)} \leq \frac{m_{r-1}}{m_{r-1}(Z_K - E)}, \tag{7.21}$$

then $m_s \geq m_s(\bar{Z}_i)$ for all $s \leq r$.

Proof. We give the proof of the first statement, the second one is similar. We need to prove the inequality $m_s \geq m_s(\bar{Z}_i)$ for $s > r+1$ as the cases $s = r, r+1$ are assumed. By lemma 5.3.5, it is enough to prove $m_s > \bar{r}_i m_s(Z_K - E)$ (note that we can have $\varepsilon_{i,n_s} \neq 0$ only if $\bar{r}_i m_{n_s}(Z_K - E) \in \mathbb{Z}$). But this follows by using lemma 7.3.9 iteratively to find

$$\bar{r}_i \leq \frac{m_{r+1}}{m_{n_{r+1}}(Z_K - E)} < \frac{m_{r+2}}{m_{n_{r+2}}(Z_K - E)} < \ldots < \frac{m_j}{m_{n_j}(Z_K - E)}.$$

□

7.5.3 Lemma. *Assume that a face $F_{n_0} \subset \Gamma(f)$ is a central trapezoid with a nondegenerate arm $n_1, \ldots, n_j$ in the direction of the x_1 axis as in 7.3.1. Let $p_1 \in F_{n_0}$ be one of the endpoints of the segment $F_{n_0} \cap F_{n_1}$, and let $p_2 \in F_{n_0}$ be the closest integral point on the adjacent boundary segment. Then the vector $p_2 - p_1$ has nonnegative x_2 and x_3 coordinates.*

Proof. We can assume that p_1 is on the x_1x_2 coordinate hyperplane. Then $\ell_3(p_1) = 0$, thus $\ell_3(p_2 - p_1) = \ell_{p_2} \geq 0$. Take the remaining vertices $p_3, p_4 \in F_{n_0}$ so that $[p_1, p_4] = F_{n_0} \cap F_{n_1}$. By the same argument as above, we then have $\ell_2(p_3 - p_4) \geq 0$.

If the segment $[p_1, p_4]$ is a top edge, then $[p_2, p_3]$ is a bottom edge, and so we have $p_2 - p_3 = a(p_1 - p_4)$ for some integer $a > 0$. Thus, $\ell_2(p_2 - p_1) = \ell_2(p_4 - p_1) + \ell_2(p_3 - p_4) + \ell_2(p_2 - p_3) \geq (a-1)\ell_2(p_1 - p_4) = (a-1)\ell_2(p_1) \geq 0$.

If $[p_1, p_4]$ is not the top edge, then the top edge is either $[p_3, p_4]$ or $[p_1, p_2]$. In either case, these two edges are parallel, and so $\ell_2(p_2 - p_1)$ and $\ell_2(p_3 - p_4)$ have the same sign and the result follows since we already proved $\ell_2(p_3 - p_4) \geq 0$. □

7.5.4 Lemma. *If $(\bar{Z}_i, E_{\bar{v}(i)}) > 0$, then $S_i = \emptyset$.*

Proof. If $l \in S_i$, then $l \geq \bar{Z}_i|_{\mathcal{N}}$ and $m_{\bar{v}(i)}(l) = m_{\bar{v}(i)}(Z_i)$ and $\mathcal{S}_Z(l) \neq \emptyset$. By lemma 7.1.6, we then have $\psi(l) \in \mathcal{S}_Z(l)$, and so $\psi(l) \geq \bar{Z}_i$, by lemma 7.1.9. We get $(\psi(l), E_{\bar{v}(i)}) \geq (\bar{Z}_i, E_{\bar{v}(i)}) > 0$, a contradiction. □

7.6 Case: $\bar{v}(i)$ is central

In this section we will assume that $\bar{v}(i)$ is a central node. We will use the notation given in proposition 3.6.4(i).

7.6.1. Assume that $\Gamma(f)$ has three nondegenerate arms. For $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, define an element $l_k \in V_Z^{\mathcal{N}}$ as follows. Require $m_{n_0^1}(l_k) = m_{n_0^1}(\bar{Z}_i)$ and $m_{n_1^\kappa}(l_k) = \min_{p\in\bar{P}_i} \ell_{n_1^\kappa}(p) + k_\kappa \alpha_{n_0^\kappa, n_1^\kappa}$. Furthermore, require that for each $\kappa = 1, 2, 3$, the sequence $m_{n_r^\kappa}(l_k)$ is an arm sequence. Since $\beta_{n_0^\kappa, n_1^\kappa} \ell_{n_0^\kappa} + \ell_{n_1^\kappa}|_{\mathbb{Z}^3} \equiv 0 \,(\mathrm{mod}\, \alpha_{n_0^\kappa, n_1^\kappa})$, we have $\beta_{n_0^\kappa, n_1^\kappa} m_{n_0^\kappa}(\bar{Z}_i) + \min_{p\in\bar{P}_i} \ell_{n_1^\kappa}(p) \equiv 0 \,(\mathrm{mod}\, \alpha_{n_0^\kappa, n_1^\kappa})$. Thus, l_k with the required properties exists and is unique by lemma 7.3.2. Now, by lemma 7.1.2 and remark 7.3.3, we find that if $l \in S_i$, then there is a k so that $l = l_k$. Indeed, we find $k_\kappa = (m_{n_1^\kappa}(l) - \min_{p\in\bar{P}_i} \ell_{n_1^\kappa}(p))/\alpha_{n_0^\kappa, n_1^\kappa}$.

In the case when $\Gamma(f)$ has two nondegenerate arms, define $l_k \in V_Z^{\mathcal{N}}$ for $k \in \mathbb{Z}^2$ as above, and similarly for $k = k_1 \in \mathbb{Z}$ if Γ contains a single nondegenerate arm.

7.6.2 Lemma. *We have $l_k \geq \bar{Z}_i$ if and only if $k \geq 0$, that is, $k_\kappa \geq 0$ for $\kappa = 1, 2, 3$.*

Proof. Using lemma 7.4.1 and lemma 5.1.5 we find

$$
\begin{aligned}
m_{n_1^\kappa}(l_k) = &\min_{p\in\bar{P}_i} \Big(\alpha_{n_0^\kappa, n_1^\kappa} (\ell_{u_{n_0^\kappa, n_1^\kappa}}(p) + k_\kappa) - \beta_{n_0^\kappa, n_1^\kappa} \ell_{n_0^\kappa}(p) \Big) \\
= &\alpha_{n_0^\kappa, n_1^\kappa} (m_{u_{n_0^\kappa, n_1^\kappa}}(\bar{Z}_i) + k_\kappa) - \beta_{n_0^\kappa, n_1^\kappa} m_{n_0^\kappa}(\bar{Z}_i) \\
= &\alpha_{n_0^\kappa, n_1^\kappa} \left\lceil \frac{m_{n_1^\kappa}(\bar{Z}_i) + \beta_{n_0^\kappa, n_1^\kappa} m_{n_0^\kappa}(\bar{Z}_i)}{\alpha_{n_0^\kappa, n_1^\kappa}} \right\rceil - \beta_{n_0^\kappa, n_1^\kappa} m_{n_0^\kappa}(\bar{Z}_i) + \alpha_{n_0^\kappa, n_1^\kappa} k_\kappa
\end{aligned}
$$

and so $m_{n_1^\kappa}(l_k) \geq m_{n_1^\kappa}(\bar{Z}_i)$ if and only if $k_\kappa \geq 0$.

Now, assuming $k \geq 0$, we get $l_k \geq \bar{Z}_i$ from lemma 7.5.2. $\square$

7.6.3 Lemma. *With l_k as above, we have $(\psi(l_k), E_{\bar{v}(i)}) = (\bar{Z}_i, E_{\bar{v}(i)}) + \sum_\kappa k_\kappa$.*

Proof. By construction we have $m_{n_1^\kappa}(l_k) = \ell(p_\kappa) + k_\kappa \alpha_{n_0^\kappa, n_1^\kappa}$ for each κ, where $p_\kappa \in \bar{P}_i$ minimizes $\ell_{n_1^\kappa}$. Therefore $m_{u_{n_0^\kappa, n_1^\kappa}}(\psi(l_k)) = \ell_{u_{n_0^\kappa, n_1^\kappa}}(p_\kappa) + k_\kappa = m_{u_{n_0^\kappa, n_1^\kappa}}(\bar{Z}_i) + k_\kappa$. Furthermore, if $e \in \mathcal{E}_{\bar{v}(i)}$, then $m_{u_e}(\psi(l_k)) = m_{u_e}(\bar{Z}_i)$ by lemma 7.1.8 and lemma 5.1.5. Therefore, $(l_k, E_{\bar{v}(i)}) = (\bar{Z}_i, E_{\bar{v}(i)}) + \sum_\kappa k_\kappa$. $\square$

7.6.4 Lemma. *Assume that $\bar{v}(i)$ is a central node and that $(\bar{Z}_i, E_{\bar{v}(i)}) = 0$. Then the set S_i consists of a single element l satisfying $z_l^{\mathcal{N}} = 1$.*

Proof. By 7.6.1 and lemma 7.6.2, we have $l = l_k$ for some $k \geq 0$ if $l \in S_i$. By lemma 7.6.3, we have $(l_k, E_{\bar{v}(i)}) = \sum_\kappa k_\kappa$, so $z_{l_k, \bar{v}(i)}^{\mathcal{N}} = 0$ unless $k = 0$ by lemma 7.1.12. Thus, to prove the lemma, we must show that, indeed, $l_0 \in S_i$. For this, we must show $z_{l_0, n_{j^c}^c}^{\mathcal{N}} = 1$ for all c. By theorem 5.3.2, we have $|\bar{P}_i| = 1$, let p be the unique point in $\bar{P}_i$. Let L_s^κ be the lines associated with the arm data $m_{n_s^\kappa}(l_0)$ for any κ. Then $p \in L_0^\kappa$, thus $z_{l_0, n_{j^\kappa}^\kappa}^{\mathcal{N}} = 1$ by lemma 7.3.8. $\square$

7.6.5 Lemma. *Assume that $F_{\bar{v}(i)}$ is a trapezoid and that $\bar{P}_i \neq \emptyset$. If k is as in 7.6.1 with $k \geq 0$, then $z_{l_k, n_{j^\kappa}^\kappa}^{\mathcal{N}} = 1$ if $j^\kappa > 0$.*

Proof. Let $L_0^\kappa, \ldots, L_{j^\kappa - 1}^\kappa$ be the lines associated with the arm sequence $m_{n_0^\kappa}(l_k), \ldots, m_{n_{j^\kappa}^\kappa}(l_k)$. Let p_1 be one of the endpoints of the segment $F_{n_0^\kappa} \cap F_{n_1^\kappa}$, and p_2 the closest integral point to p_1 on the adjacent edge of $F_{n_0^\kappa}$ with endpoint p_1. Take $p \in \bar{P}_i$ so that $m_{n_1^\kappa}(l_k) = \ell_{n_1^\kappa}(p)$ and set $p_0 = p + k_\kappa(p_2 - p_1)$. Take $\kappa', \kappa'' \in \mathbb{Z}$ so that $\{\kappa, \kappa', \kappa''\} = \{1, 2, 3\}$. By lemma 7.5.3, p_0 has nonnegative $x_{\kappa'}$ and $x_{\kappa''}$ coordinates. Furthermore, $p_0 \in L_0^\kappa$ because $\ell_{n_0^\kappa}(p_2 - p_1) = 0$ and $\ell_{n_1^\kappa}(p_2 - p_1) = \alpha_{n_0^\kappa, n_1^\kappa}$. The lemma now follows from lemma 7.3.8. $\square$

7.6.6 Lemma. *If $\bar{v}(i)$ is a central node, then* $\sum_{l\in S_i} = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)})+1\}$.

Proof. The case when $(\bar{Z}_i, E_{\bar{v}(i)}) \geq 0$ is covered by lemmas 7.5.4 and 7.6.4. We start by showing that if $F_{\bar{v}(i)}$ is a triangle, then this is indeed the case. We have $(\bar{Z}_i, E_{\bar{v}(i)}) \geq \bar{r}_i((Z_K - E), E_{\bar{v}(i)}) = -\bar{r}_i$, because $\bar{Z}_i \geq \bar{r}_i(Z_K - E)$ and $m_{\bar{v}(i)}(\bar{Z}_i) = \bar{r}_i m_{\bar{v}(i)}(Z_K - E)$. If $\bar{r}_i < 1$, then the statement follows. If $\bar{r}_i = 1$, then $\bar{P}_i \subset (F_{\bar{v}(i)} - (1,1,1)) \cap \mathbb{Z}^3_{\geq 0} = \emptyset$ and so $(\bar{Z}_i, E_{\bar{v}(i)}) > 1$ by theorem 5.3.2.

We therefore assume that $F_{\bar{v}(i)}$ is a trapezoid and that $(\bar{Z}_i, E_{\bar{v}(i)}) < 0$. In that case, if $k \geq 0$, we have $z^{\mathcal{N}}_{l_k,n} = 1$ for all $\mathcal{N} \ni n \neq \bar{v}(i)$. Writing $n = n^\kappa_r$ with $r > 0$, this follows from construction if $r < j^\kappa$, and from lemma 7.6.5 if $r = j^\kappa$. We therefore have $S_i = \left\{ l_k \,\middle|\, k \geq 0,\, z^{\mathcal{N}}_{l_k,\bar{v}(i)} \neq 0 \right\}$.

If $\Gamma(f)$ has exactly one nondegenerate arm, then $\mathcal{N}_{\bar{v}(i)} = \{n^1_1\}$. By lemmas 7.6.3 and 7.1.12, we have $z^{\mathcal{N}}_{l_k,\bar{v}(i)} = 1$ if $k \leq (-\bar{Z}_i, E_{\bar{v}(i)})$, and $z^{\mathcal{N}}_{l_k,\bar{v}(i)} = 0$ otherwise. Therefore,

$$\sum_{l\in S_i} z^{\mathcal{N}}_l = \left|\left\{ k \in \mathbb{Z} \,\middle|\, 0 \leq k \leq (-\bar{Z}_i, E_{\bar{v}(i)}) \right\}\right| = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}.$$

If $\Gamma(f)$ has two nondegenerate arms, then, for $k = (k_1, k_2)$, we have $z^{\mathcal{N}}_{l_k,\bar{v}(i)} = 1$ if $k_1 + k_2 = (-\bar{Z}_i, E_{\bar{v}(i)})$ and $z^{\mathcal{N}}_{l_k} = 0$ otherwise. Therefore,

$$\sum_{l\in S_i} z^{\mathcal{N}}_l = \left|\left\{ k \in \mathbb{Z}^2_{\geq 0} \,\middle|\, k_1 + k_2 = (-\bar{Z}_i, E_{\bar{v}(i)}) \right\}\right| = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}.$$

If $\Gamma(f)$ has three nondegenerate arms, then we have $z^{\mathcal{N}}_{l_k,\bar{v}(i)} = 1$ if $\sum_\kappa k_\kappa = (-\bar{Z}_i, E_{\bar{v}(i)})$, $z^{\mathcal{N}}_{l_k,\bar{v}(i)} = -1$ if $\sum_\kappa k_\kappa = (-\bar{Z}_i, E_{\bar{v}(i)}) - 1$ and $z^{\mathcal{N}}_{l_k} = 0$ otherwise. Therefore,

$$\begin{aligned}
\sum_{l\in S_i} z^{\mathcal{N}}_l &= \left|\left\{ k \in \mathbb{Z}^3_{\geq 0} \,\middle|\, \sum_\kappa k_\kappa = (-\bar{Z}_i, E_{\bar{v}(i)}) \right\}\right| \\
&\quad - \left|\left\{ k \in \mathbb{Z}^3_{\geq 0} \,\middle|\, \sum_\kappa k_\kappa = (-\bar{Z}_i, E_{\bar{v}(i)}) - 1 \right\}\right| \\
&= \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}.
\end{aligned}$$

□

7.6.7 Remark. It is simple to verify that in the case when $\Gamma(f)$ has three nondegenerate arms, then, for each $p \in \bar{P}_i$, there is a unique element $l_p \in S_{i,p}$ and that $z^{\mathcal{N}}_{l_p} = 1$. Therefore eqs. (7.18) and (7.19) do indeed hold in this case. This is, however, not generally true in the case when $\Gamma(f)$ contains a trapezoid, and only one or two arms.

7.7 Case: One or two nondegenerate arms

In this subsection we assume that the diagram $\Gamma(f)$ has one or two nondegenerate arms. We will assume given a fixed step i in the computation sequence and that $\bar{v}(i)$ is not the central node.

7.7.1 Lemma. *We have* $\sum_{l\in S_i} z^{\mathcal{N}}_l = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)})\}$.

Proof. This will follow from lemmas 7.7.2 and 7.7.3 and lemmas 7.4.3 and 7.4.4, as well as theorem 5.3.2. □

7.7.2 Lemma. *Assume that the diagram $\Gamma(f)$ contains exactly one nondegenerate arm and that $F_{\bar{v}(i)}$ is not a central face. Then, for each $p \in \bar{P}_i$, the set $S_{i,p}$ contains a unique element l_p and $z^{\mathcal{N}}_{l_p} = 1$.*

Proof. As in 7.3.1, assume that $\mathcal{N} = \{n_\nu, \ldots, n_j\}$. We can then assume that $\bar{v}(i) = n_r$ for some $r > 0$. Fix a $p \in \bar{P}_i$. If $r > \nu$, let $m_\nu, \ldots, m_j$ be the arm sequence constructed in lemma 7.4.5, with the requirement $m_r = \ell_{n_r}(p)$ and $m_{r-1} = \ell_{n_{r-1}}(p)$. If $r = \nu = 1$, let $m_1, \ldots, m_j$ be the arms sequence defined by requiring $m_1 = \ell_{n_1}(p)$ and $m_2 = \ell_{n_2}(p)$, which exists and is unique by lemma 7.3.2. We then have an element $l_p \in V^{\mathcal{N}}_Z$ with $m_{n_s}(l) = m_s$ for all s.

The inequality $l_p \geq \bar{Z}_i$ follows from lemma 7.5.2.

Let L_s for $s = \nu, \ldots, j-1$ be the lines associated with the arm sequence $m_\nu, \ldots, m_j$. We then have $p \in L_{r-1}$ if $r > \nu$ and $p \in L_r$ if $r < j$. By lemma 7.3.8 we therefore get $z^{\mathcal{N}}_{l_p, n_j} = 1$. In order to show $z^{\mathcal{N}}_{l_p, n_\nu} = 1$, we must, by lemma 7.1.12, prove $(\psi(l_p), E_{n_\nu}) \leq 0$. We have $m_{n_\nu}(l_p) = \ell_{n_\nu}(p)$ by lemma 7.4.5. Since $-b_{n_\nu}\ell_{n_\nu}(p) + \sum_{u \in \mathcal{V}_{n_\nu}} \ell_u(p) = 0$, it is enough to show $m_u(\psi(l)) \leq \ell_u(p)$ for $u \in \mathcal{V}_{\bar{v}(i)}$. In the case $u = u_{n_\nu, n_{\nu+1}}$, we have $m_u = \beta_{n_\nu, n_{\nu+1}} m_{n_\nu} + m_{n_{\nu+1}} = \beta_{n_\nu, n_{\nu+1}} \ell_{n_\nu}(p) + \ell_{n_{\nu+1}}(p) = \ell_u(p)$ by lemma 7.1.2 and the definition of ℓ_u. If, however, $u = u_e$ for some $e \in \mathcal{E}_{n_\nu}$, then

$$m_u(\psi(l)) = \left\lceil \frac{\beta_e m_{n_0}}{\alpha_e} \right\rceil \leq \frac{\beta_e \ell_{n_0}(p) + \ell_{n_e^*}(p)}{\alpha_e} = \ell_u(p).$$

by lemma 7.1.8 and the fact that $\beta_e \ell_{n_0}(p) + \ell_{n_e^*}(p) \equiv 0 \pmod{\alpha_e}$ and $\ell_{n_e^*}(p) \geq 0$ since $p \in \mathbb{Z}^3_{\geq 0}$. □

7.7.3 Lemma. *Assume that the diagram $\Gamma(f)$ contains exactly two nondegenerate arms and that $\bar{v}(i)$ is not a central face. Then, for each $p \in \bar{P}_i$, the set $S_{i,p}$ contains a unique element l_p and $z^{\mathcal{N}}_{l_p} = 1$.*

Proof. Use the notation given in proposition 3.6.4. We can then assume that $\bar{v}(i) = n^1_r$ for some $r \geq 1$. Similarly as above, using lemma 7.3.2, we find numbers $m^1_0, \ldots, m^1_{j^1} \in \mathbb{Z}$ so that if $l \in S_{i,p}$, then $m_{n^1_r}(l) = m^1_r$ for $0 \leq r \leq j^1$. Furthermore, by lemma 7.4.5, we have $m^1_s = \ell_{n^1_s}(p)$ for $s \leq r+1$. If $\Gamma(f)$ contains a central edge, let c be the number of central edges. If $\Gamma(f)$ contains a central node, set $c = 0$. In either case, we have $n^1_s = n^2_{c-s}$ for $s \leq c$. Note that in the case of a central edge, we can assume $j^\kappa > c - 1$ for $\kappa = 1, 2$, since otherwise the statement is covered by lemma 7.7.2. In particular, we have nodes $n^1_0, n^2_0 \in \mathcal{N}$.

In the case of a central edge, we therefore have $m_{n^2_s}(l) = m^1_{c-s}$ for $s = 0, 1$, for all $l \in S_{i,p}$. Let $m^2_0, \ldots, m^2_{j^2}$ be the arm sequence with $m^2_0 = m^1_c$ and $m^2_1 = m^2_{c-1}$. Then, for any $l \in S_{i,p}$, we have $m_{n^2_s}(l) = m^2_s$.

In the case of a central node, we have a number $m^2_1 \in \mathbb{Z}$, uniquely determined by the equation

$$\frac{m^2_1 + \beta_{n^2_0, n^2_1} m^1_0}{\alpha_{n^2_0, n^2_1}} - b_{n^1_0} m^1_0 + \frac{\beta_{n^1_0, n^1_1} m^1_0 + m^1_1}{\alpha_{n^1_0, n^1_1}} + \sum_{e \in \mathcal{E}_{n^1_0}} \frac{\beta_e m^1_0}{\alpha_e} = 0.$$

Setting $m^2_0 = m^1_0$, lemma 7.3.2 determines an arm sequence m^2_s with $m_{n^2_s}(l) = m^2_s$ for all $0 \leq s \leq j^2$ and $l \in S_{i,p}$.

We define $l_p \in V^{\mathcal{N}}_Z$ by $m_{n^e_s}(l_p) = m^e_s$. We have proved that if $l \in S_{i,p}$, then $l = l_p$. To prove the lemma, we must show that indeed, $l_p \in S_{i,p}$. For this, we need to prove that $z^{\mathcal{N}}_{l_p,n^e_{je}} = 1$ for $e = 1, 2$ and that $l_p \geq \bar{Z}_i$. As in the case of a single nondegenerate arm, we find $z^{\mathcal{N}}_{l_p,n^1_{j1}} = 1$, and $m_{n^1_s}(l_p) \geq m_{n^1_s}(\bar{Z}_i)$ for all s.

Let L^κ_s be the lines associated with the arm sequence $m^\kappa_0, \ldots, m^\kappa_{j^\kappa}$. In the case when $\Gamma(f)$ contains a central edge, note that $L^2_0 = L^1_{c-1}$. In particular, $p \in L^2_0$, and so $z^{\mathcal{N}}_{l_p,n^2_{j2}} = 1$ by lemma 7.3.8. It is also clear that $m^2_s \geq m_{n^2_s}(\bar{Z}_i)$ for $s = c, c-1$ and that

$$\frac{\ell_{n^2_{c-1}}(p)}{m_{n^2_{c-1}}(Z_K - E)} \leq \frac{\ell_{n^2_c}(p)}{m_{n^2_c}(Z_K - E)}$$

since $p \in C_{n^2_r}$. Therefore, by lemma 7.5.2, we have $m_{n^2_s}(l_p) \geq m_{n^2_s}(\bar{Z}_i)$ for $s \geq c - 1$, hence $l_p \geq \bar{Z}_i$.

Next we consider the case when $\Gamma(f)$ contains a central node. We need to prove $m^2_s \geq m_{n^2_s}(\bar{Z}_i)$ for $s \geq 1$ and $z^{\mathcal{N}}_{l_p,n^2_{j2}} = 1$. The former follows in a similar way as above as soon as we prove

$$\frac{m^2_0}{m_{n^2_0}(Z_K - E)} \leq \frac{m^2_1}{m_{n^2_1}(Z_K - E)}. \tag{7.22}$$

Comparing the two equations

$$\frac{m^2_1 + \beta_{n^2_0,n^2_1} m^2_0}{\alpha_{n^2_0,n^2_1}} - b_{n^1_0} m^1_0 + \frac{\beta_{n^1_0,n^1_1} m^1_0 + m^1_1}{\alpha_{n^1_0,n^1_1}} + \sum_{e \in \mathcal{E}_{n^1_0}} \left\lceil \frac{\beta_e m^1_0}{\alpha_e} \right\rceil = 0$$

and

$$\frac{\ell_{n^2_1} + \beta_{n^2_0,n^2_1} \ell_{n^2_0}(p)}{\alpha_{n^2_0,n^2_1}}(p) - b_{n^1_0} \ell_{n^1_0} + \frac{\beta_{n^1_0,n^1_1} \ell n^1_0(p) + \ell n^1_1(p)}{\alpha_{n^1_0,n^1_1}} + \sum_{e \in \mathcal{E}_{n^1_0}} \frac{\beta_e \ell_{n^1_0}(p) + \ell_{n^*_e}(p)}{\alpha_e} = 0,$$

and the fact that $\ell_{n^1_0}(p) = m^1_0 = m^2_0$ and $\ell_{n^1_1}(p) = m^1_1$ we find $m^2_1 \geq \ell_{n^2_1}(p)$, hence

$$\frac{m^2_0}{m_{n^2_0}(Z_K - E)} \leq \frac{\ell_{n^2_0}(p)}{m_{n^2_0}(Z_K - E)} \leq \frac{\ell_{n^2_1}(p)}{m_{n^2_1}(Z_K - E)} \leq \frac{m^2_1}{m_{n^2_1}(Z_K - E)},$$

proving eq. 7.22. We observe from these equations that we also have $m^2_1 \equiv \ell_{n^2_1}(p) \pmod{\alpha_{n^2_0,n^2_1}}$

Finally, we will prove $z^{\mathcal{N}}_{l_p,n^2_{j2}} = 1$. By lemma 7.3.8, it is enough to prove that the line L^2_0 contains a point with nonnegative x_1 and x_3 coordinates.

We start with the case when $F_{n^1_0}$ is a trapezoid. Let p_1 be one of the endpoints of the segment $F_{n^2_0} \cap F_{n^2_1}$, and p_2 the closest integral point on an adjacent boundary segment of $F_{n^2_0}$. By lemma 7.5.3, the vector $p_2 - p_1$ has nonnegative x_1 and x_3 coordinates. Since $m_1^2 \geq \ell_{n_1^2}(p)$, as we proved above, the same holds for the point

$$p_0 = p + \frac{m_1^2 - \ell_{n_1^2}(p)}{\alpha_{n_0^2, n_1^2}}(p_2 - p_1)$$

which is an integral point by our previous observation. Since $\ell_{n_0^2}(p_2 - p_1) = 0$ and $\ell_{n_1^2}(p_2 - p_1) = \alpha_{n_0^2, n_1^2}$, by proposition 3.2.10, we find $p_0 \in L_0^2$.

Next, we will prove $z^{\mathcal{N}}_{l_p, n^2_{j^2}} = 1$, assuming that $F_{n_0^1}$ is a central triangle. Define a point p_0 by requiring $\ell_n(p) = m_n(l_p)$ for $n = n_1^1, n_0^1, n_1^2$. Using the same proof as in lemma 7.3.6, we see that p_0 exist, is unique, and $p_0 \in \mathbb{Z}^3$. By definition, we also have $p_0 \in L_0^1$ and $p_0 \in L_1^2$, so, as in the previous case, it suffices to show that $\ell_1(p_0) \geq 0$ and $\ell_3(p_0) \geq 0$. Since $F_{n_0^1}$ is a central triangle, and $\Gamma(f)$ has one degenerate arm, we have $|\mathcal{E}_{n_0^1}| = 1$. Let $e \in \mathcal{E}_{n_0^1}$ be the unique element in this set. We then have

$$m_{u_e}(l_p) = \left\lceil \frac{\beta_e m_{n_0^1}(l_p)}{\alpha_e} \right\rceil \leq \frac{\beta_e \ell_{n_0^1}(p) + \ell_{n_e^*}(p)}{\alpha_e} = \ell_{u_e}(p).$$

Furthermore, subtracting eq. 3.1 (with $n = n_0^1$), evaluated at p_0, from $(l_p, E_{n_0^2}) = 0$, we get $m_{u_e}(l_p) = \ell_{u_e}(p_0)$, thus, $\ell_{u_e}(p - p_0) \geq 0$. Evaluating eq. 3.1 at $p - p_0$ gives $\ell_{u_e}(p - p_0) + \ell_{n_1^2}(p - p_0)/\alpha_{n_0^2, n_1^2} = 0$ and so $\ell_{n_1^2}(p - p_0) \leq 0$.

Give names q_1, q_2, q_3 to the vertices of the triangle $F_{n_0^1}$ as in fig. 5, that is, q_1 lies on the $x_2 x_3$ axis, etc. By definition, we have $p, p_0 \in L_0^1$. Furthermore, the line L_0^1 is parallel to the primitive vector $q_2 - q_3$. Therefore, there is a $k \in \mathbb{Z}$ so that $p_0 = p + k(q_2 - q_3)$. By convexity of $\Gamma_+(f)$ we have $\ell_{n_1^2}(q_2 - q_3) \geq 0$. Therefore, by the previous inequality, we get $k \geq 0$. Since $\ell_3(q_2 - q_3) = \ell_3(q_2) \geq 0$, we have $\ell_3(p_0) \geq 0$. If $k = 0$, then $p_0 = p$, and we get $\ell_1(p_0) \geq 0$. Otherwise, we have $p_0 = p + k(q_2 - q_3)$ with $k > 0$. Since $p \in (\cup_{r=1}^{j^1} C_{n_1^r} \cap \Gamma_-(f)) - (1,1,1)$, we have $\ell_2(q_3 - q_2) > \ell_2(p)$, therefore, $\ell_2(p_0) < 0$. Since the arm in the direction of the x_3 axis is assumed degenerate, we have $a, b \in \mathbb{Z}_{>0}$ so that $\ell_{n_e^*} = a\ell_1 + b\ell_2$. Furthermore, we have

$$\ell_{n_e^*}(p_0) = \alpha_e \ell_{n_0^1}(p_0) - \beta_e \ell_{u_e}(p_0) = \left\lceil \frac{\beta_e m_{n_0^1}(l_p)}{\alpha_e} \right\rceil - \beta_e m_{n_0^1}(l_p) \geq 0.$$

All this gives $\ell_1(p_0) \geq 0$, finishing the proof. □

7.8 Case: Three nondegenerate arms

In this subsection we will assume that the diagram $\Gamma(f)$ contains a central node and three nondegenerate arms. We will assume given a fixed step i in the computation sequence and that $\bar{v}(i)$ is not the central node.

7.8.1. We use the notation introduced in proposition 3.6.4(i). We can assume that for some $1 \leq r \leq j^1$ we have $\bar{v}(i) = n_r^1$. By lemmas 7.4.3 and 7.4.4, we have $S_i' = \emptyset$, so in order to prove eq. 7.1, it is enough to prove $\sum_{l \in S_{i,p}} z_l^{\mathcal{N}} = 1$ for all $p \in \bar{P}_i$. For $\kappa = 2, 3$, define

$$S_{i,p}^{\kappa} = \left\{ l \in S_{i,p} \,\middle|\, m_{n_1^\kappa}(l) < \ell_{n_1^\kappa}(p) \right\}$$

and set $S_{i,p}^0 = S_{i,p} \setminus (S_{i,p}^2 \cup S_{i,p}^3)$.

7.8.2 Lemma. *We have* $\sum_{l\in S_i} z_l^{\mathcal{N}} = \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)})\}$.

Proof. This will follow from lemmas 7.8.7, 7.8.11 and 7.8.13 and theorem 5.3.2. $\square$

7.8.3. By lemma 7.3.2, there is an arm sequence $m_0^1, \dots, m_{j^1}^1$ satisfying $m_{r-1}^1 = \ell_{n_{r-1}^1}(p)$ and $m_r^1 = \ell_{n_r^1}(p)$, and we have $m_{n_s^1}(l) = m_s^1$ for all $l \in S_{i,p}$. Fix an $l \in S_{i,p}$. We then have $m_{n_1^2}(l) \equiv -\beta_{n_0^2,n_1^2} m_{n_0^2}(l) = -\beta_{n_0^2,n_1^2}\ell_{n_0^2}(p) \equiv \ell_{n_1^2}(p) \,(\mathrm{mod}\,\alpha_{n_0^2,n_1^2})$, and so there is a $k\in\mathbb{Z}$ so that $m_{n_1^2}(l) = \ell_{n_1^2}(p) + k\alpha_{n_0^2,n_1^2}$. Using the equation

$$-b_{n_0^1}\ell_{n_0^1} + \sum_{\kappa=1}^{3} \frac{\beta_{n_0^\kappa,n_1^\kappa}\ell_{n_0^\kappa} + \ell_{n_1^\kappa}}{\alpha_{n_0^\kappa,n_1^\kappa}} + \sum_{e\in\mathcal{E}_{n_0^1}} \frac{\beta_e \ell_{n_0^1} + \ell_{n_e^*}}{\alpha_e} = 0$$

and the fact that $(\psi(l), E_{n_0^1}) =: \eta \in \{0,-1\}$ by lemma 7.1.12, we find that

$$\frac{\ell_{n_1^3}(p) - m_1^{3,k,\eta}}{\alpha_{n_0^3,n_1^3}} = k - \eta + \sum_{e\in\mathcal{E}_{n_0^1}} \left\lceil \frac{\beta_e m_{n_0^1}}{\alpha_e} \right\rceil - \frac{\beta_e \ell_{n_0^1}(p) + \ell_{n_e^*}(p)}{\alpha_e}. \tag{7.23}$$

where $m_1^{3,k,\eta} = m_{n_1^3}(l)$. Note that since $(\beta_e\ell_{n_0^1} + \ell_{n_e^*})/\alpha_e$ is an integral functional, the summand corresponding to $e\in\mathcal{E}_{n_0^1}$ on the right in eq. 7.23 is integral. Since $\ell_{n_0^1}(p) = m_{n_0^1}$ and $\ell_{n_e^*}(p) \geq 0$, each such summand is < 1. Thus, it follows that these summands are nonpositive.

7.8.4 Definition. For $k\in\mathbb{Z}$, define $m_0^{2,k} = m_0^1$ and $m_1^{2,k} = \ell_{n_1^2}(p) + k\alpha_{n_0^2,n_1^2}$. Furthermore, for $\eta = 0,-1$, let $m_0^{3,k,\eta} = m_0^1$ and define $m_1^{3,k,\eta}$ as the unique solution to eq. 7.23. Then, by lemma 7.3.2, there exist unique arm sequences $(m_s^{2,k})_{s=0}^{j^2}$ and $(m_s^{3,k,\eta})_{s=0}^{j^3}$ with the given first two initial terms. Define $l_p^{k,\eta} \in V_Z^{\mathcal{N}}$ by $m_{n_s^1}(l_p^{k,\eta}) = m_s^1$ for $0\leq s\leq j^1$, $m_{n_s^2}(l_p^{k,\eta}) = m_s^{2,k}$ for $0\leq s\leq j^2$ and $m_{n_s^3}(l_p^{k,\eta}) = m_s^{3,k,\eta}$ for $0\leq s\leq j^3$.

In 7.8.3 we have thus proven

7.8.5 Lemma. *If $l\in S_{i,p}$, then $l = l_p^{k,\eta}$ for some $\mathbb{Z}$ and $\eta\in\{0,-1\}$. In fact, we have*

$$S_{i,p} = \left\{ l_p^{k,\eta} \,\middle|\, l_p^{k,\eta} \geq \bar{Z}_i,\; z^{\mathcal{N}}_{l_p^{k,\eta},n_{j^2}^2} = z^{\mathcal{N}}_{l_p^{k,\eta},n_{j^3}^3} = 1 \right\}.$$

$\square$

7.8.6 Definition. Let $k_0\in\mathbb{Z}$ be the unique number so that $m_1^{3,k_0,0} = \ell_{n_1^3}(p)$.

It is clear from the remark after eq. 7.23 that $k_0 \geq 0$.

7.8.7 Lemma. *We have $S_{i,p}^2 \cap S_{i,p}^3 = \emptyset$.*

Proof. If $l_p^{k,\eta}\in S_{i,p}^2\cap S_{i,p}^3$, then, by definition, $k<0$ and $k > k_0 + \eta \geq 0$. This is clearly impossible. $\square$

7.8.8 Lemma. *We have $m_{n_s^1}(l_p^{k,\eta}) \geq m_{n_s^1}(\bar{Z}_i)$ for $0\leq s\leq j^1$ and $z^{\mathcal{N}}_{l_p^{k,\eta},n_{j^1}^1} = 1$ for any k,η.*

Proof. This follows in exactly the same way as the corresponding statement in the proof of lemma 7.7.2. ☐

7.8.9 Lemma. *If $k \geq 0$ then $m_{n_s^2}(l_p^{k,\eta}) \geq m_{n_s^2}(\bar{Z}_i)$ for $0 \leq s \leq j^2$. Similarly, if $k \leq k_0 + \eta$, then $m_{n_s^3}(l^{k,\eta}) \geq m_{n_s^3}(\bar{Z}_i)$ for $0 \leq s \leq j^3$.*

Proof. We prove the statement for the second arm. The statement for the third arm follows similarly. If $k \geq 0$, then

$$\frac{m_{n_1^2}(l_p^{k,\eta})}{m_{n_1^2}(Z_K - E)} \geq \frac{\ell_{n_1^2}(p)}{m_{n_1^2}(Z_K - E)} \geq \frac{\ell_{n_0^2}(p)}{m_{n_0^2}(Z_K - E)} = \frac{m_{n_0^2}(l_p^{k,\eta})}{m_{n_0^2}(Z_K - E)},$$

since $p \in \mathbb{R}^3_{\geq 0} \setminus \cup_{r=1}^{j^2} C_{n_r^2}$ (as in the proof of lemma 7.3.9). Thus, the result follows from lemma 7.5.2. ☐

7.8.10 Lemma. *(i) If $F_{n_0^1}$ is a trapezoid and $k \geq 0$, then $z^{\mathcal{N}}_{l_p^{k,\eta}, n_{j^2}^2} = 1$.*

(ii) If $F_{n_0^1}$ is a trapezoid and $k \leq k_0 + \eta$, then $z^{\mathcal{N}}_{l_p^{k,\eta}, n_{j^3}^3} = 1$.

(iii) If $F_{n_0^1}$ is a triangle, then $z^{\mathcal{N}}_{l_p^{0,0}, n_{j^2}^2} = 1$ and $z^{\mathcal{N}}_{l_p^{0,0}, n_{j^3}^3} = 1$.

Proof. We start by proving (i), the proof of (ii) is similar. Let p_1 be one of the endpoints of the segment $F_{n_0^2} \cap F_{n_0^2}$ and p_2 the closest integral point to p_1 on the adjacent boundary segment of $F_{n_0^2}$. Let $p_0 = p + k(p_2 - p_1)$. By proposition 3.2.10 we then have $\ell_n(p_0) = m_n(l_p^{k,\eta})$ for $n = n_0^2, n_1^2$. Thus, the result follows from lemma 7.3.8

(iii) follows in a similar way, since $m_n(l_p^{0,0}) = \ell_n(p)$ for $n = n_0^2, n_1^2, n_1^3$. ☐

7.8.11 Lemma. *We have $\sum_{l \in S_{i,p}^{\kappa}} z_l^{\mathcal{N}} = 0$ for $\kappa = 2, 3$.*

Proof. We prove the lemma for $\kappa = 2$, the case $\kappa = 3$ follows similarly.

For any $l \in S_{i,p}^2$ we have $z_l^{\mathcal{N}} = \pm 1$. In fact, there are $k \in \mathbb{Z}_{<0}$ and $\eta \in \{0, -1\}$ so that $l = l_p^{k,\eta}$. Then $z_l^{\mathcal{N}} = (-1)^\eta$. Therefore, the lemma is proved as soon as we prove that for any $k \in \mathbb{Z}_{<0}$ we have $l_p^{k,0} \in S_{i,p}^2$ if and only if $l_p^{k,-1} \in S_{i,p}^2$. Now, $m_{n_s^2}(l_p^{k,0}) = m_{n_s^2}(l_p^{k,-1})$ for all k. In particular we have $z^{\mathcal{N}}_{l_p^{k,0}, n_{j^2}^2} = z^{\mathcal{N}}_{l_p^{k,-1}, n_{j^2}^2}$. Furthermore, by lemma 7.8.9, we have $m_{n_s^3}(l_p^{k,\eta}) \geq m_{n_s^3}(\bar{Z}_i)$ for any $k < 0$ (since $k_0 \geq 0$). It therefore suffices to prove that if $z^{\mathcal{N}}_{l_p^{k,0}, n_{j^2}^2} = 1$ then $z^{\mathcal{N}}_{l_p^{k,0}, n_{j^3}^3} = 1$ if and only if $z^{\mathcal{N}}_{l_p^{k,-1}, n_{j^3}^3} = 1$ for all $k < 0$. But this follow immediately from lemma 7.8.12 ☐

7.8.12 Lemma. *If $F_{n_0^1}$ is a triangle and $\{a, b, c\} = \{1, 2, 3\}$, define the points $q_1, q_2, q_3 \in \mathbb{Z}^3$ as the vertices of $F_{n_0^1}$ so that q_a, q_b are the end points of the segment $F_{n_0^c} \cap F_{n_1^c}$.*

(i) If $F_{n_0^1}$ is a trapezoid, then $z^{\mathcal{N}}_{l_p^{k,\eta}, n_{j^3}^3} = 1$ for any $k < 0$ and $\eta \in \{0, -1\}$.

(ii) If $F_{n_0^1}$ is a triangle and either $\ell_2(p) \geq \ell_2(q_2)$ or $\ell_1(q_2) \leq \ell_1(q_3)$, then $z^{\mathcal{N}}_{l_p^{k,\eta}, n_{j^3}^3} = 1$ for any $k < 0$ and $\eta \in \{0, -1\}$.

(iii) If $F_{n_0^1}$ is a triangle, $\ell_2(p) < \ell_2(q_2)$, and $\ell_1(q_2) \geq \ell_1(q_3)$, then then $z^{\mathcal{N}}_{l_p^{k,\eta}, n_{j^2}^2} = 0$ for any $k < 0$ and $\eta \in \{0, -1\}$.

Proof. Take $k \in \mathbb{Z}_{<0}$ and $\eta \in \{0, -1\}$. Let $L_0^3, \ldots, L_{j^3-1}^3$ be the lines associated with the arm sequence $m_{n_0^3}(l_p^{k,\eta}), \ldots, m_{n_{j^3}^3}(l_p^{k,\eta})$.

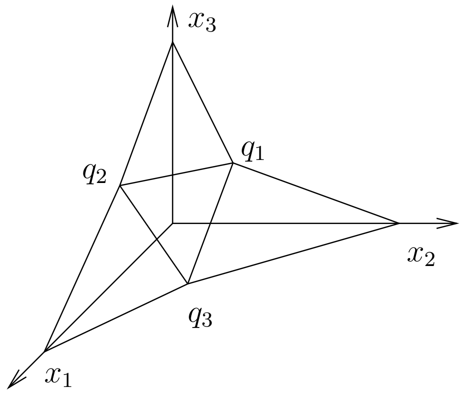

Figure 5: q_1, q_2, q_3 are the vertices of the triangle $F_{n_0^1}$.

(i) Define p_0 by $p_0 = p + (k_0 + \eta - k)(q_3 - q_1)$. Using lemma 7.5.3, and the fact that $k_0 + \eta - k \geq 0$, we find that p_0 has nonnegative x_1 and x_2 coordinates. Furthermore, proposition 3.2.10 gives $\ell_{n_0^3}(p_0) = m_{n_0^3}(l_p^{k,\eta})$ and $\ell_{n_1^3}(p_0) = m_{n_1^3}(l_p^{k,\eta})$, that is, $p_0 \in L_0^3$. The result now follows from lemma 7.3.8.

(ii) As in the previous case, the result will follow as soon as we prove that L_0^3 contains an integral point with nonnegative x_1 and x_2 coordinates.

First, we assume $\ell_2(p) \geq \ell_2(q_2)$. Let $A = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \subset \mathbb{R}^3$. We want to show $L_0 \cap A \cap \mathbb{Z}^3 \neq \emptyset$. Now, (for the purposes of this proof only) let π^3 be the canonical projection from $\mathbb{R}^3$ to the x_1x_2 plane. Furthermore, let ℓ be a linear function on the x_1x_2 plane so that $\ell(\pi(q_1)) = \ell(\pi(q_2)) > 0$. It is then clear that $\ell(\pi(q_3)) > \ell(\pi(q_2))$. If we define p_0 by the same method as in the previous case, it is not necessarily true that $p_0 \in A$. We see, however, that $\ell(\pi(p_0)) \geq \ell(\pi(p))$. Let $L \in \mathbb{R}^3$ be the line which is parallel to L_0^3 and passes through p. We find that the segment $\pi(L) \cap A$ is longer than the segment $\pi(L_s^3) \cap A$. This implies that the segment $L_0^3 \cap A$ is longer than the segment $L \cap A$. Now, the segment $L \cap A$ contains p, as well as $p + q_1 - q_2$, by hypothesis, and so has length at least one. Thus, $L_0^3 \cap A$ has length at least one as well. But a segment of length at least one contains an integral point.

Now, if $\ell_1(q_2) \geq \ell_1(q_3)$, then we can proceed in a similar fashion as in (i). Indeed, if we define $p_0 = p + k(q_3 - q_2)$, then, by our assumptions, we find that p_0 has nonnegative x_1 and x_3 coordinates. Furthermore, we have $p_0 \in L_0^3$, and so the result follows from lemma 7.3.8.

(iii) In this case, let $L_0^2, \ldots, L_{j^3}^2$ be the lines associated with the arms sequence $m_{n_0^2}(l_p^{k,\eta}), \ldots, m_{n_{j^2}^2}(l_p^{k,\eta})$. Using proposition 3.2.10, we find that $p_0 = p + k(q_2 - q_3) \in L_0^2$. The vector $q_1 - q_3$ is primitive and we have $\ell_{n_0^2}(q_1 - q_3) = \ell_{n_1^2}(q_1 - q_3) = 0$. Thus, $L_0^2 \cap \mathbb{Z}^3 = \{p_0 + h(q_1 - q_3) \,|\, h \in \mathbb{Z}\}$. It is clear that

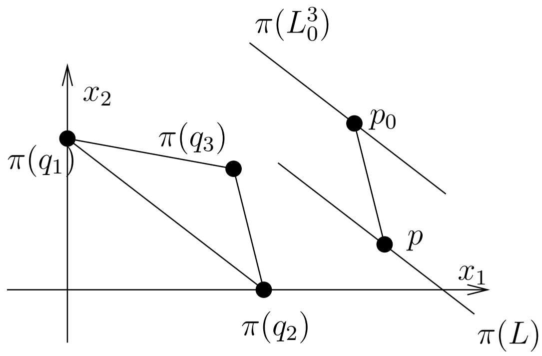

Figure 6: A projection.

$\ell_3(q_2) > \ell_3(p)$ and $\ell_3(q_3) = 0$. Since $k < 0$, we get $\ell_3(p_0) < 0$, so we find $\ell_3(p_0 + h(q_1 - q_3)) < 0$ for all $h \geq 0$. If, however, $h > 0$, then

$$\begin{aligned}
\ell_1(p_0 + h(q_1 - q_3)) &= \ell_1(p_0 + (q_1 - q_3)) + (h-1)\ell_1(q_1 - q_3) \\
&\leq \ell_1(p_0 + (q_1 - q_3)) \\
&= \ell_1(p + k(q_2 - q_3) + (q_1 - q_3)) \\
&= \ell_1(p + (k+1)(q_2 - q_3) + (q_1 - q_2)) \\
&= \ell_1(p + (q_1 - q_2)) + \ell_1((k+1)(q_2 - q_3)) \\
&< 0.
\end{aligned}$$

Here we use both assumptions in the last inequality. We have thus proved that no integral point in the line L_0^2 has nonnegative x_1 and x_3 coordinates. By lemma 7.3.8 we get $z^{\mathcal{N}}_{l_p^{k,\eta}, n^2_{j^2}} = 0$. $\square$

7.8.13 Lemma. *We have* $\sum_{l \in S^0_{i,p}} z_l^{\mathcal{N}} = 1$.

Proof. By definition, and lemma 7.8.5, we have

$$S^0_{i,p} \subset \{l_p^{0,0}, \ldots, l_p^{k_0,0}, l_p^{0,-1}, \ldots, l_p^{k_0-1,-1}\}. \tag{7.24}$$

Since $z^{\mathcal{N}}_{l_p^{k,\eta}, n_0^1} = (-1)^\eta$ by lemma 7.1.12, the lemma is proved as soon as we prove equality in eq. 7.24.

In the case of a triangle, it follows from definition that $k_0 = 0$. Therefore, lemma 7.8.10 shows that for any element l of the right hand side of eq. 7.24, we have $z^{\mathcal{N}}_{l,n^2_{j^2}} = z^{\mathcal{N}}_{l,n^3_{j^3}} = 1$. Furhtermore, lemma 7.8.9 shows that for such an l we have $l \geq \bar{Z}_i$. Thus, $l \in S_{i,p}$ and so equality holds in eq. 7.24. $\square$

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List of symbols

$(\bar{Z}_i)$ Subsequence of computation sequence, p. 43

$(\cdot)_\lambda$ Generalized eigenspace, p. 22

$(\cdot,\cdot)$ Intersection form, p. 9

$\square_{l,I}$ Cube, p. 17

$\wedge$ Meet operation, p. 10

$\alpha(a,b)$ Determinant of vectors a, b, p. 26

$A_{\mathbb{C}^3}$ Graded associated with Newton filtration, p. 32

$\mathcal{A}_{\mathbb{C}^3}(r)$ Newton filtration on $\mathcal{O}_{\mathbb{C}^3,0}$, p. 32

α_e α_{n_e,n_e^*}, p. 34

$\alpha_{n,n'}$ Determinant of ℓ_n and $\ell_{n'}$, p. 26

α_s $\alpha_{n_s,n_{s+1}}$, p. 58

A_X Graded associated with Newton filtration, p. 32

$\mathcal{A}_X(r)$ Newton filtration on $\mathcal{O}_{X,0}$, p. 32

$-b_v$ Selfintersection number, p. 7

$\beta(a,b)$ Numerator of vectors a, b, p. 26

β_e β_{n_e,n_e^*}, p. 34

$\beta_{n,n'}$ Denominator of ℓ_n and $\ell_{n'}$, p. 26

β_s $\beta_{n_s,n_{s+1}}$, p. 58

$\overline{\beta}_s$ $\beta_s^{-1} \pmod{\alpha_s}$, p. 58

$\chi(l)$ Analytic euler characteristic extended to L, p. 17

CIC Casson Invariant Conjecture, p. 15

C_n Cone associated with n, p. 44

c_{rF^-} Content of diluted polygon with boundary conditions, p. 37

c_S Content of segment, p. 36

δ_v Degree of a vertex, p. 8

$\mathrm{div}(g)$ Divisorial valuation of g, p. 30

E Exceptional divisor of a resolution, p. 8

$\mathcal{E}$ Ends, p. 34

$\mathcal{E}_{n_j}^\lambda$ Subset of $\mathcal{E}_{n_j}$, p. 35

$\mathcal{E}_n$ Ends of a node n, p. 34

E_v^* Dual cycles, p. 9

$\mathrm{eu}(\mathbb{H}^*(A,w))$ Normalized Euler characteristic, p. 18

$\varepsilon_{v,i}$ Boundary values at step i, p. 44

E_v Irreducible component of exceptional divisor, p. 8

φ Newton weight function, p. 32

$\mathcal{F}(l)$ Divisorial ideal, p. 12

F_i^{cn} Cone section, p. 47

$F_i^{\mathrm{cn}-}$ Cone section with boundary conditions, p. 47

f_F Principal part, p. 25

$F_. H^2(Y_\infty, \mathbb{C})$ Hodge filtration, p. 22

F_n Compact face of Newton polygon, p. 26

$F_n^{\mathrm{nb}}(Z)$ Polygon associated with n and Z, p. 31

G Resolution or plumbing graph, p. 8

$\Gamma(f)$ Newton diagram, p. 25

G^* Extended graph, p. 26

$\Gamma_+(f)$ Newton polytope, p. 25

$F_n(Z)$ Face associated with n and Z, p. 31

$\Gamma_-(f)$ Under Newton diagram, p. 25

$\Gamma_+(Z)$ Newton polytope associated with Z, p. 31

H Cokernel of the intersection form, p. 9

$H(t)$ Hilbert series, p. 12

$\mathbb{H}^*(A,w)$ Lattice cohomology, p. 18

$\mathbb{H}^*_{\mathrm{red}}(A,w)$ Reduced lattice cohomology, p. 18

$\hat{H}$ Pontrjagin dual of H, p. 11

h_l Coefficient of Hilbert series, p. 12

$h^{p,q}$ Hodge numbers, p. 23

$h^{p,q}_\lambda$ Equivariant Hodge numbers, p. 23

$H_v^=(Z)$ Hyperplane associated with v and Z, p. 31

$H_v^{\geq}(Z)$ Halfspace associated with v and Z, p. 31

I	Intersection matrix, p. 7
$I^{-1}_{v,w}$	Entries of the inverted intersection matrix, p. 9
I	Intersection matrix, p. 9
K	Canonical cycle, p. 9
$\mathrm{KS}_M(\sigma, g, \eta)$	Kreck–Stolz invariant, p. 14
L	Intersection lattice, p. 9
$\lambda(M)$	Casson–Walker–Lescop invariant of M, p. 14
ℓ_1	Standard coordinate function in $\mathbb{R}^3$, p. 27
$\lambda_C(M)$	Casson invariant of M, p. 14
$\lambda_{CW}(M)$	Casson–Walker invariant of M, p. 14
$\lambda_{CWL}(M)$	Casson–Walker–Lescop invariant of M, p. 14
ℓ_f	Newton weight function, p. 31
ℓ_n	Support function of Newton polygon, p. 26
$l\|_{\mathcal{N}}$	Reduction to nodes, p. 54
L'	Dual intersection lattice, p. 9
ℓ_{rS}	Support function of a diluted polygon, p. 36
ℓ_S	Support function of an integral polygon, p. 36
L_s	Associated lines, p. 60
μ	Milnor number, p. 23
M	Link of a singularity, p. 7
$m_v(Z)$	Coefficient of a cycle Z, p. 10
$\mathcal{N}$	Index set for compact two dimensional faces, p. 26
ν	Either 0 or 1, p. 58
$\mathcal{N}^*$	Index set for two dimensional faces, p. 26
π	Resolution $\pi : \tilde{X} \to X$, p. 8
$P(t)$	Poincaré series, p. 12
$P^{\mathcal{A}}_{\mathbb{C}^3}(t)$	Poincaré series associated with Newton filtration, p. 32
$P^{\mathcal{A}}_X(t)$	Poincaré series associated with Newton filtration, p. 32
$\mathrm{pc}\, P(t)$	Periodic constant of $P(t)$, p. 19
p_g	Geometric genus, p. 12

$\bar{P}_i$ Set of integral point at step i, p. 43

p_l Coefficient of Poincaré series, p. 12

$P^{\mathrm{neg}}(t)$ Negative part of $P(t)$, p. 19

$P^{\mathrm{pol}}(t)$ Polynomial part of $P(t)$, p. 19

p_s Associated vertices, p. 60

$\psi(l)$ Section to $\pi^{\mathcal{N}}$, p. 55

$\mathcal{Q}$ Set of cubes, p. 17

$Q(t)$ Equivariant counting function, p. 11

$Q_0(t)$ Counting function, p. 11

$q_{l'}$ Coefficient of (equivariant) counting function, p. 11

$\bar{r}_i$ Ratio at step i, p. 44

σ_{can} Canonical spin$^{\mathrm{c}}$ structure on link, p. 14

S_i Set of $l \in L^{\mathcal{N}}$ contributing to the sum $q_{Z_{i+1}} - q_{Z_i}$, p. 57

S'_i Subset of S_i, p. 64

$S_{i,p}$ Subset of S_i, p. 64

$S^{\kappa}_{i,p}$ Subset of S_i, p. 70

$\mathrm{Sp}(f,0)$ Spectrum, p. 22

$\mathrm{Sp}_I(f,0)$ Part of spectrum, p. 23

$\mathrm{Spin}^{\mathrm{c}}(E)$ Set of spin$^{\mathrm{c}}$ structures on the vector bundle E, p. 13

$\mathrm{Spin}^{\mathrm{c}}(M)$ Set of spin$^{\mathrm{c}}$ structures on the manifold M, p. 13

$\mathcal{S}_{\mathrm{top}}$ Lipman cone, topological semigroup, p. 10

$\mathcal{S}'_{\mathrm{top}}$ Lipman cone, topological semigroup, p. 10

$\mathbf{sw}^0_M(\sigma)$ Seiberg–Witten invariant, p. 14

SWIC Seiberg–Witten Invariant Conjecture, p. 15

$\mathcal{S}_Z$ $\mathcal{S}_{\mathrm{top}} \cap V_Z$, p. 55

$\mathcal{S}_Z(l)$ $\mathcal{S}_Z \cap (\pi^{\mathcal{N}})^{-1}(l)$, p. 55

$\mathcal{S}^{\mathcal{N}}_Z$ $\pi^{\mathcal{N}}(\mathcal{S}_Z)$, p. 55

$\mathcal{T}^0_{M,\sigma}$ Normalized Reidemeister–Turaev torsion, p. 14

$\mathcal{T}_{M,\sigma}$ Reidemeister–Turaev torsion, p. 14

$t_{n,n'}$ Length of segment $F_n \cap F_{n'}$, p. 26

u_e Neighbour of node on leg containing end e, p. 34

$u_{n,n'}$ Neighbour of n on branch containing n', p. 27

$\mathcal{V}$ Vertex set of a resolution or plumbing graph, p. 7

$\mathcal{V}^*$ Vertex set of G^*, p. 26

$\mathcal{V}_v^*$ Set of neighbours in G^*, p. 26

V_Z $V_Z' \cap L$, p. 54

V_Z' $\mathbb{Z}\langle E_v^* | v \in \mathcal{N} \cup \mathcal{E}\rangle$, p. 54

w Weight function, p. 18

$W.H^2(Y_\infty, \mathbb{Q})$ Weight filtration, p. 22

$\mathrm{wt}(g)$ Weight of g, p. 30

$(X, 0)$ Germ of a singular space, p. 7

$\tilde{X}$ Resolution of X, p. 8

$x(Z)$ Laufer closure operator, p. 39

$Z(t)$ Equivariant zeta function, p. 11

$Z_0(t)$ Zeta function, p. 11

Z_K Anticanonical cycle, p. 9

$z_{l'}$ Coefficient of (equivariant) zeta function, p. 11

$z_{l',v}$ Factor of coefficient of zeta function, p. 56

$z_l^{\mathcal{N}}$ Coefficient of reduced zeta function, p. 56

$z_{l,n}^{\mathcal{N}}$ Factor of coefficient of reduced zeta function, p. 56

$Z_{\min}$ Artin's minimal cycle, p. 10

$Z_0^{\mathcal{N}}(t)$ Reduced zeta function, p. 56

$\mathbb{Z}[[t^L]]$ Set of power series, p. 11

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