EXTREMAL PROBLEMS ON FINITE SETS AND POSETS

by

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Contents

Abstract

The overarching theme of the thesis is the investigation of extremal problems involving forbidden partially ordered sets (posets). In particular, we will be concerned with the function $\mathrm{La}(n, P)$, defined to be the maximum number of sets we can take in the Boolean lattice $2^{[n]}$ without introducing the relations of a poset P as containment relations among the sets. This function plays an analogous role in the setting of nonuniform hypergraphs (set systems) as the extremal function $\mathrm{ex}(n, G)$ in graph theory and has already been studied extensively. This type of extremal problem was formally introduced in 1983 by Katona and Tarján, and there have been more than 50 papers on the subject. The majority of the results involve bounding $\mathrm{La}(n, P)$ both in a general sense, as well as for numerous specific cases of posets.

The thesis is divided into 5 main chapters. The first chapter gives a summary of the history of forbidden poset problems as well as the relevant background information from extremal set theory.

In the second chapter we give a significant improvement of the general bounds on $\mathrm{La}(n, P)$ as a function of the height of the poset $h(P)$ and the size of the poset $|P|$ due to Burcsi and Nagy and later Chen and Li. The resulting bound is in a certain sense best possible. We also give an improvement of the bound on the so-called Lubell function in the induced version of the problem, and we introduce a new chain counting technique which may have additional applications. The results in this chapter were joint work with Dániel Grósz and Abhishek Methuku.

The third chapter introduces a new partitioning technique on cyclic permutations. In this chapter we find a surprising generalization of a result of De Bonis, Katona and Swanepoel on a poset known as the butterfly. Namely, we show that one can introduce a subdivision to one of the edges of the Hasse diagram of this poset and prove that nonetheless the same bound holds on its extremal number. Using the new partitioning technique, we also determine the exact bound for the La function of an infinite class of pairs of posets. The results in this chapter were joint work with Abhishek Methuku.

In the fourth chapter we introduce a new variation on the extremal subposet problem. Namely, we assume that the family must also be intersecting. We give a novel generalization of the partition method of Griggs and Li and prove an exact bound for intersecting, butterfly-free families. We

also give a new proof of a result of Gerbner on intersecting k-Sperner families and determine the equality cases for the first time. The results in this chapter were joint work with Dániel Gerbner and Abhishek Methuku.

In the fifth chapter we prove a new version of the De Bruijn Erdős theorem for partially ordered sets, as well as another version in a graph setting. The results in this subsection were joint work with Pierre Aboulker, Guillaume Lagarde, David Malec and Abhishek Methuku.

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Chapter 1

Introduction

Most of the results in this thesis are on extremal problems for finite sets. More specifically, the problems we consider involve maximizing the number of sets we can take in a collection $\mathcal{F} \subseteq 2^{[n]}$, provided that if we view $\mathcal{F}$ as a partially ordered set (poset) with respect to the inclusion relation $\subseteq$, then $\mathcal{F}$ does not contain a given poset P as a subposet (induced or noninduced).

In the following subsections we first give an overview of the terminology and basic results from extremal set theory which we will need. We then move on to survey the history of forbidden poset problems. Furthermore, we describe the essential techniques developed so far.

1.1 Background on extremal set theory

The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$, and for any set X, the power set of X is denoted by 2^X. Sets $\mathcal{F} \subseteq 2^{[n]}$ are referred to as set families, hypergraphs, or collections of sets. The collection of all subsets of $[n]$ which have size k is denoted $\binom{[n]}{k}$ and is referred to as the k^{th} level of $2^{[n]}$. A hypergraph $\mathcal{F}$ is called r-uniform if every set in $\mathcal{F}$ has cardinality r. For an r-uniform hypergraph $\mathcal{F}$, we denote the *shadow* and *shade* of $\mathcal{F}$ respectively by

$$\Delta\mathcal{F} = \{G : |G| = r - 1 \text{ and there exists } F \in \mathcal{F} \text{ such that } G \subset F\};$$
$$\nabla\mathcal{F} = \{G : |G| = r + 1 \text{ and there exists } F \in \mathcal{F} \text{ such that } F \subset G\}.$$

We will consider problems where the goal is to maximize $|\mathcal{F}|$ over all $\mathcal{F} \subseteq 2^{[n]}$ subject to various constraints. Among the most fundamental results in extremal set theory is the famous theorem of

Sperner [75]. A collection of sets $\mathcal{A}$ is said to be an antichain (or Sperner family) if there do not exist two sets A and B in $\mathcal{A}$ such that $A \subset B$. Sperner proved the following:

Theorem 1 (Sperner [75]). *Let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Moreover, equality holds if and only if $\mathcal{A}$ is equal to a middle level (i.e., $\mathcal{A} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\binom{[n]}{\lceil n/2 \rceil}$).

We recall Sperner's proof since we use some ideas from it later:

Proof (Sperner). Suppose that for every level $i < \lceil n/2 \rceil$ we had an injection ϕ_i mapping the entire level $\binom{[n]}{i}$ into the level $\binom{[n]}{i+1}$ such that for all F of size i, $F \subset \phi_i(F)$. We will show how we could use these injections to modify an arbitrary antichain $\mathcal{A}$ to one whose size we can easily bound. Let $i^* < \lceil n/2 \rceil$ be the smallest cardinality of a set contained in $\mathcal{A}$. Since $\mathcal{A}$ is an antichain we have, in particular, that every set of size $i^* + 1$ which contains a set from $\mathcal{A}$ is not in $\mathcal{A}$. For each i, let $\mathcal{A}_i = \{A \in \mathcal{A} : |A| = i\}$. Then, if we replace $\mathcal{A}_{i^*}$ with $\phi_{i^*}(\mathcal{A}_{i^*})$, no new containment relation will be introduced. Indeed, there are no sets of size smaller than $i^* + 1$, and any larger set which contains a set $A \in \phi_{i^*}(\mathcal{A}_{i*})$ already contains a set in $\mathcal{A}$. The cardinality of the new family $\mathcal{A} \cup \phi_{i^*}(\mathcal{A}_{i^*}) \setminus \mathcal{A}_{i^*}$ is the same as $\mathcal{A}$, since ϕ_{i^*} is an injection. Thus, by successively applying injections $\phi_{i^*}, \phi_{i^*+1} \ldots \phi_{\lceil n/2 \rceil - 1}$ we find a new antichain of the same size with no set of size smaller than $\lceil n/2 \rceil$. We will now show that such injections exist by Hall's theorem.

Let $i < \lceil n/2 \rceil$ and $\mathcal{B}$ be an arbitrary collection of sets of size i. It is enough to show $|\nabla \mathcal{B}| \geq |\mathcal{B}|$. To do this we count the pairs (B, B') such that $B \in \mathcal{B}$, $|B'| = i + 1$ and $B \subset B'$ in two different ways. On the one hand, each $B \in \mathcal{B}$ has $n - i$ sets of size $i + 1$ containing it so the number of pairs is exactly $|\mathcal{B}|(n - i)$. On the other hand, each $B' \in \nabla \mathcal{B}$ contains $i + 1$ sets of size i including all $B \in \mathcal{B}$ contained in B' and possibly some sets which are not in $\mathcal{B}$. Thus, $|\nabla \mathcal{B}|(i + 1) \geq |\mathcal{B}|(n - i)$. Rearranging yields

$$|\nabla \mathcal{B}| \geq \frac{n-i}{i+1} |\mathcal{B}| \geq |\mathcal{B}|,$$

since $i < \lceil n/2 \rceil$. It follows that Hall's condition is satisfied for each $i < \lceil n/2 \rceil$ yielding the existence of the injections ϕ_i. A completely analogous argument shows, for $i > \lceil n/2 \rceil$, there exist injections ψ_i mapping the level $\binom{[n]}{i}$ into the level $\binom{[n]}{i-1}$ such that $B \supset \psi_i(B)$ for any set B of cardinality i.

Thus, for each antichain $\mathcal{A}$ we can find an antichain of the same size consisting only of sets of size $\lceil n/2 \rceil$ and so $|\mathcal{A}| \leq \binom{n}{\lceil n/2 \rceil} = \binom{n}{\lfloor n/2 \rfloor}$. □

Sperner's theorem is the starting point for a wide variety of research directions in extremal set theory. The next subsection is entirely devoted to one such direction, which is the main topic of the present thesis. First, however, we consider some other important extensions of this theorem as well as some additional proof techniques which we will need.

In solving a number theoretic problem of Littlewood and Offord, Erdős [28] introduced a generalization of Sperner's theorem to the case where, instead of forbidding just one comparable pair of sets A and B in a set family $\mathcal{A}$, one forbids a $(k+1)$-tuple of sets $A_1, A_2, \ldots, A_{k+1}$ such that $A_1 \subset A_2 \subset \ldots \subset A_{k+1}$. Such a $(k+1)$-tuple is referred to as a $(k+1)$-*chain*. Families of sets with no $(k+1)$-chain are commonly referred to as k-*Sperner*. For notational convenience, we will use the notation $\Sigma(n,k)$ to denote the sum of the k largest binomial coefficients of the form $\binom{n}{i}$ where $0 \leq i \leq n$. Equivalently,

$$\Sigma(n,k) = \sum_{i=\lfloor \frac{n-k+1}{2} \rfloor}^{\lfloor \frac{n+k-1}{2} \rfloor} \binom{n}{i}.$$

Erdős proved

Theorem 2 (Erdős [28])**.** *Let $\mathcal{A} \subseteq 2^{[n]}$ be a k-Sperner family, then*

$$|\mathcal{A}| \leq \Sigma(n,k).$$

We don't reproduce Erdős's proof as considerably shorter proofs exist. However, we mention that the technique used by Erdős was similar to that of Sperner. The argument again finds a sequence of k-Sperner families which are increasingly central. But, rather than looking at just consecutive levels, one must use Menger's theorem to produce long disjoint chains of consecutive sets in $2^{[n]}$ and move the sets in $\mathcal{A}$ along these chains towards the middle levels. In his paper, Erdős also gave a simple proof for a special case of this theorem. Namely, every collection of sets without a pair of sets A and B such that $A \subset B$ and $|B| - |A| \geq k$ has size at most $\Sigma(n,k)$. This is indeed a special case of Erdős's theorem on k-Sperner families because the difference in the sizes of the largest and smallest set in any $(k+1)$-chain is at least k. The dual problem, forbidding $|B| - |A| \leq k$ was solved by Katona [46] using a variation of symmetric chain decompositions.

Perhaps the most well-known generalization of Sperner's theorem is due to Lubell [60], Yamamoto [78], Meshalkin [64] and Bollobás [11]. They proved the following inequality, which became known as the LYM-inequality after three of its discoverers (Bollobás and Meshalkin actually proved stronger results):

Theorem 3 (LYM-Inequality [60, 78, 64, 11]). *Let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain, then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Equality holds if and only if $\mathcal{A}$ is a level in $2^{[n]}$.

Theorem 3 is easily seen to imply Sperner's theorem because if $\mathcal{A}$ is an antichain we have

$$\frac{|\mathcal{A}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Lubell's proof of this theorem uses a famous double counting argument involving pairs $(A, \mathcal{C})$ consisting of a set and a maximal chain (a chain consisting of a set of each possible size). It is interesting that, in fact, this sort of double counting argument dates back even earlier to an inequality of Kraft [53] in the theory of prefix-free codes.

This theorem has a very natural generalization to the k-Sperner case (which was first considered by Katona in [44]).

Theorem 4 (Katona [44]). *Let $\mathcal{A} \subseteq 2^{[n]}$ be a k-Sperner family, then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq k. \tag{1.1}$$

Equality holds if and only if $\mathcal{A}$ is the union of k levels in $\mathcal{A}$.

Observe that if we know Theorem 3 is true, then we can deduce Theorem 4 in a very simple manner. Indeed, let $\mathcal{A}$ be k-Sperner and take the following decomposition of $\mathcal{A}$ originally considered by Mirsky [68] by defining for $1 \leq i \leq k$:

$$\mathcal{A}_i = \{A : A \in \mathcal{A} \text{ and the longest chain in } \mathcal{A} \text{ with maximal set } A \text{ has length } i\}.$$

It is clear that the $\mathcal{A}_i$ are antichains and partition $\mathcal{A}$. Then, by Theorem 3 we have

$$\sum_{A \in \mathcal{A}_i} \frac{1}{\binom{n}{|A|}} \leq 1$$

for all i, and summing over i yields the result (equality cases included). This technique is used several times in a more complicated way in Chapter 3 of this thesis.

We now present a generalization of Lubell's proof to the k-chain setting (as in [44]). We also give a proof of the equality cases which we haven't found in the literature, but which has some similarity to a proof by Lovász of the equality case of Sperner's theorem [57].

Proof. Let $\mathcal{A}$ be a k-Sperner family. Call a chain $\mathcal{C}$ maximal if it contains a set of every level in $2^{[n]}$. We will count pairs $(A, \mathcal{C})$ where $A \in \mathcal{A}$, $\mathcal{C}$ is a maximal chain and $A \in \mathcal{C}$. On the one hand, each A is contained in precisely $|A|\,!\,(n-|A|)!$ maximal chains $\mathcal{C}$. Indeed, to count how many maximal chains go through A, we can start with A and determine in how many ways we can build a chain downward from A to $\varnothing$ and in how many ways we can build a chain upward from A to $[n]$. The downward portion consists of sequentially removing elements of A, and this can be done in $|A|\,!$ ways. The upward part of the chain consists of adding elements from the complement of A one by one, which we can do in $(n-|A|)!$ ways. Thus, we have that the number of pairs $(A, \mathcal{C})$ is

$$\sum_{A \in \mathcal{A}} |A|\,!\,(n-|A|)!\,.$$

On the other hand, if we first fix a maximal chain $\mathcal{C}$, then $\mathcal{C}$ can contain at most k sets from $\mathcal{A}$ because $\mathcal{A}$ is k-Sperner. Thus, the number of pairs $(A, \mathcal{C})$ is at most $kn!$. Combining, we get

$$\sum_{A \in \mathcal{A}} |A|\,!\,(n-|A|)! \leq kn!\,,$$

and rearranging we obtain

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq k,$$

as desired. Equality implies we must have k sets from $\mathcal{A}$ in every maximal chain. Suppose we have a family satisfying equality with some but not all sets from level i. Say A is an i-element set in our collection and B is an i-element set outside of our collection. Then, by deleting elements of $A \setminus B$

one at a time and replacing them by elements of $B \setminus A$ we eventually get two sets C and D where $C \in \mathcal{A}, D \notin \mathcal{A}$ and $|C \cap D| = i - 1$. Consider the maximal chain

$$\mathcal{C} = A_0 \subset A_1 \subset \ldots \subset A_{i-2} \subset C \cap D \subset D \subset C \cup D \subset A_{i+2} \subset \ldots \subset A_n,$$

where the A_i's are chosen arbitrarily to form a maximal chain. Equality in the (1.1) implies that we have k sets in $\mathcal{A} \cap \mathcal{C}$. This, however, is a contradiction because $D \notin \mathcal{A}$, but every other set in the chain is comparable to C. That is, we must have exactly k sets in $\mathcal{A} \cap \mathcal{C}$ all of which are also all comparable to $C \in \mathcal{A}$, thus yielding a $k+1$ chain. $\square$

The left-hand side of (1.1) is often referred to as the *Lubell function.* We will now show that one can deduce Erdős's result in a simple way from Theorem 4.

Lemma 1. *If $\mathcal{A} \subset 2^{[n]}$ satisfies*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq k, \tag{1.2}$$

then $|\mathcal{A}| \leq \Sigma(n,k)$.

Proof. Suppose that $\mathcal{A} \subseteq 2^{[n]}$ satisfies (1.2). Let $B_1, B_2, \ldots, B_{n+1}$ denote the binomial coefficients of the form $\binom{n}{i}$ in decreasing order. By contradiction, suppose that $|\mathcal{A}| > B_1 + \cdots + B_k$, and write $|\mathcal{A}| = B_1 + \cdots + B_k + \alpha$ where $\alpha > 0$. Let $b_1, \ldots, b_{|\mathcal{A}|}$ denote the $|\mathcal{A}|$ smallest numbers of the form $\frac{1}{\binom{n}{|F|}}$, where $F \in 2^{[n]}$, in decreasing order. Then b_1 through b_{B_1} are equal to $1/B_1$, b_{B_1+1} through $b_{B_1+B_2}$ are equal to $1/B_2$ and so on. Thus, we have

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \geq b_1 + \cdots + b_{|\mathcal{A}|} \geq B_1 \frac{1}{B_1} + \cdots + B_k \frac{1}{B_k} + \alpha \frac{1}{B_{k+1}} > k,$$

a contradiction. $\square$

A family of sets $\mathcal{A}$ is said to be *intersecting* if for all $A, B \in \mathcal{A}$ we have $A \cap B \neq \varnothing$. If for a positive integer t we have that $A, B \in \mathcal{A}$ implies $|A \cap B| \geq t$, then $\mathcal{A}$ is said to be *t-intersecting.*

By far, the most well-known extremal problem on intersecting families is the Erdős-Ko-Rado theorem [29]. For the full statement we need the notion of a star. A *star* with *kernel* x is a collection of all subsets of a given size which contain a fixed element x of the ground set. The size of a star is $\binom{n-1}{r-1}$, and it is reasonable to expect that this should be the maximal intersecting family. In fact it is, and this is the content of the theorem. We present Katona's elegant proof, which introduced the method of cyclic permutations.

Theorem 5 (Erdős-Ko-Rado [29] $t=1$ case)**.** *Let $\mathcal{A} \subset \binom{[n]}{r}, 2r \leq n$ be intersecting, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1},$$

and for $r < n/2$, equality holds if and only if $\mathcal{A}$ is a star.

Proof. A *cyclic permutation* of $[n]$ is an arrangement of the numbers 1 through n along a circle. A set A ($A \neq \varnothing$ or $[n]$) is said to be an *interval* along the cyclic permutation if the elements of A occur consecutively in this arrangement. We think of clockwise as the forward direction so that we may consider each interval as having a first and last element in a natural way. A star along σ is the collection of r intervals which contain some fixed element.

We double count pairs (A, σ) where $A \in \mathcal{A}$, σ is a cyclic permutation of $[n]$ and A is an interval along σ. For each σ there are at most r sets from $\mathcal{A}$ which are intervals along σ with equality if and only if they form a star about some element on σ. To see this, fix an interval $x_1, \ldots, x_r$ and do the following pairing off argument: for each i, $1 \leq i \leq r-1$, consider the interval which ends at x_i and the interval which begins at x_{i+1}. Since the family is intersecting and $2r \leq n$ it is clear that for each i we may take at most one interval from each pair. To obtain r intervals, we must take an interval from every one of the $r-1$ pairs.

If we take r intervals and $2r < n$, then the star structure is immediate from the fact that we cannot take an interval ending at x_i and one beginning at x_{i+2} for any i. Thus, if we ever take the interval from the pair that ends at x_i, then for $j > i$ we must also take the interval which ends at x_j rather than the one beginning at x_{j+1}. It follows immediately that we have a star (about the first x_i which is the end of an interval we take in the pair).

Since there are $(n-1)!$ cyclic permutations, the number of pairs is at most $r(n-1)!$. Fixing first a set $A \in \mathcal{A}$ there are $r!\,(n-r)!$ permutations containing A and so the number of pairs is $r!\,(n-r)!\,|\mathcal{A}|$. Thus, we have $|\mathcal{A}| \leq \binom{n-1}{r-1}$, as required.

Equality implies we have a star on every σ. We will show that this implies that our set family is also a star. Take a cyclic permutation and suppose we have a star with kernel a_r. We may take

$$\sigma = b, a_1, a_2, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots, a_{2r-1}, c_1, \ldots, c_{n-2r}.$$

Importantly, note that for $r < n/2$ there is at least one c. Now, since $\{b, a_1, \ldots, a_{r-1}\}$ is not in $\mathcal{A}$ but $\{a_1, \ldots, a_r\} \in \mathcal{A}$, we permute the elements $a_1, \ldots, a_{r-1}$ in σ and we will again have a star with kernel a_r. For any ordering of $a_1, \ldots, a_{r-1}$, we may also permute any of the elements of $[n] \setminus \{b, a_1, \ldots, a_r\}$ as we wish and still obtain a star with kernel a_r. Thus, we can already find any set F which contains a_r and does not contain b in our family. By applying the same argument to cyclic permutations

□

Note 1. *For $r = n/2$ there are several extremal families. Specifically, we can take one set from each set complement pair.*

There are a variety of proofs of the Erdős-Ko-Rado theorem, yielding generalizations in different directions. There is a proof using linear algebra given by Lovász [56] in determining the Shannon-capacity of the Kneser graph. There is a short proof due to Daykin [22] which invokes the Kruskal Katona theorem [55, 47]. There is a recent proof due to Frankl and Füredi [31] using Katona's shadow theorem on intersecting families [43].

In their original proof, Erdős, Ko and Rado proved the following general result for large n:

Theorem 6 (Erdős-Ko-Rado [29] general t)**.** *For any r and t, there exists an $n_0(r,t)$ such that if $n \geq n_0(r,t)$, then*

$$|\mathcal{A}| \leq \binom{n-t}{r-t}.$$

Wilson [77] proved the theorem for the minimal $n_0(r,t)$ for which the extremal family is still a star using linear algebra. In [5] Ahlswede and Khachatrian determined the size of the maximal t-intersecting r uniform family for all n and classified the extremal configurations.

We won't give the proof of any of these results here, but instead we mention an important variation of this theorem due to Milner [67].

Theorem 7 (Milner [67])**.** *Let $A \subseteq 2^{[n]}$ be a t-intersecting antichain, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n+t+1}{2} \rfloor}.$$

We will mainly be interested in the case $t = 1$ (this case was proved independently by Schönheim [73]).

Theorem 8 (Milner [67], Schönheim [73])**.** *Let $A \subseteq 2^{[n]}$ be an intersecting antichain, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}.$$

Simpler proofs of this case were later given by Katona [48] and Scott [74]. The theorem also follows immediately from an inequality of Greene, Katona and Kleitman [35]:

Theorem 9 (Greene, Katona, Kleitman [35])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting antichain, then*

$$\sum_{\substack{F \in \mathcal{F} \\ |F| \leq \frac{n}{2}}} \frac{1}{\binom{n}{|F|-1}} + \sum_{\substack{F \in \mathcal{F} \\ |F| > \frac{n}{2}}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

A related inequality was proved earlier by Bollobás [12]:

Theorem 10 (Bollobás [12])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting antichain, and assume that for all $F \in \mathcal{F}$ we have $|F| \leq n/2$, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-1}{|F|-1}} \leq 1.$$

The proofs of Theorem 10 and Theorem 9 use a similar idea. Here we present the proof of Theorem 10 which can be found in both [12] and [35].

Proof. As in Katona's proof of the Erdős-Ko-Rado we will double count pairs (F, σ), where σ is a cyclic permutation and $F \in \mathcal{F}$ is an interval along σ. Define the following weight function:

$$w(F, \sigma) = \begin{cases} \frac{1}{|F|}, & \text{if } F \in \mathcal{F} \text{ and } F \text{ is an interval along } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

First, suppose we fix a set $F \in \mathcal{F}$. Then, there are exactly $|F|\,!\,(n-|F|)!$ cyclic permutations that contain F as an interval, and we have

$$\sum_{F \in \mathcal{F}} \sum_{\sigma} w(F, \sigma) = \sum_{F \in \mathcal{F}} \frac{1}{|F|} |F|\,!\,(n-|F|)! = \sum_{F \in \mathcal{F}} (|F|-1)!\,(n-|F|)!\,. \tag{1.3}$$

Now, fix a cyclic permutation σ. We will use a pairing off argument similar to the proof of Theorem 5. Fix an interval, $F = x_1 x_2 \ldots x_m$, from $\mathcal{F}$ of minimal length m along σ. Since $\mathcal{F}$ is intersecting, for each $1 \le i \le m-1$ we may have an interval ending at x_i or one beginning at x_{i+1} but not both (here we use $|F| \le n/2$). Moreover, since $\mathcal{F}$ is an antichain, there can only be one such interval. Let $\mathcal{F}^\sigma$ be the family of all elements of $\mathcal{F}$ which are intervals along σ. We have

$$\sum_{\sigma}\sum_{F\in\mathcal{F}} w(F,\sigma) = \sum_{\sigma}\sum_{F\in\mathcal{F}^\sigma}\frac{1}{|F|} \le \sum_{\sigma}\sum_{F\in\mathcal{F}^\sigma}\frac{1}{m} \le \sum_{\sigma}\frac{m}{m} = (n-1)! \tag{1.4}$$

where the first inequality holds because m is the minimal length of an interval and $\frac{1}{x}$ is decreasing, and the second inequality follows from the pairing off argument described above. Comparing (1.3) and (1.4) we obtain

$$\sum_{F\in\mathcal{F}} (|F|-1)!\,(n-|F|)! \le (n-1)!\,.$$

Dividing through by $(n-1)!$ yields the desired inequality. $\square$

Both Theorems 10 and 9 are special cases of a more general result of Péter Erdős, Péter Frankl, and Gyula O.H. Katona [30] who determined the convex hull of the profile vectors which can be obtained by intersecting Sperner families. We will study generalizations of these inequalities in Chapter 4. Next, we turn our attention to the history and key ideas in forbidden poset problems.

1.2 Background on forbidden subposet problems

Given two posets P and Q, we say that P is a *subposet* of Q if there exists an injection ϕ from P to Q such that if $x \le y$ in P, then $\phi(x) \le \phi(y)$ in Q. We say that P is an *induced subposet* of Q if there exists an injection ϕ such that $x \le y$ in P if and only if $\phi(x) \le \phi(y)$ in Q. An important notational point is that in most of the literature outside forbidden poset problems, the word subposet is used for the induced case. Observe that any family $\mathcal{F} \subset 2^{[n]}$ can be regarded as a poset with respect to the containment relation $\subseteq$.

Katona and Tarján [51] initiated the investigation of the following problem: Given a poset P, how large of a collection $\mathcal{F}$ can we find in $2^{[n]}$ such that $\mathcal{F}$ does not contain P as a subposet? Analogously to the extremal function $\mathrm{ex}(n,G)$ in graph theory, the functions

$$\mathrm{La}(n,P) = \max\{\mathcal{F} : \mathcal{F} \subset 2^{[n]} \text{and } \mathcal{F} \text{ does not contain } P \text{ as a subposet}\}$$

$$\mathrm{La}^{\#}(n,P) = \max\{\mathcal{F} : \mathcal{F} \subset 2^{[n]} \text{and } \mathcal{F} \text{ does not contain } P \text{ as an induced subposet}\}$$

are the main object of study in forbidden poset problems. If we wish to forbid a pair of posets P and Q, we simply write $\mathrm{La}(n, P, Q)$ and $\mathrm{La}^{\#}(n, P, Q)$ respectively.

Let P_{k+1} be the poset on the base set $\{x_1, \ldots, x_{k+1}\}$ with relations $x_1 \le x_2 \le \ldots \le x_{k+1}$. Then, the theorems of Sperner [75] and Erdős [28] are simply the statements that $\mathrm{La}(n, P_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and $\mathrm{La}(n, P_{k+1}) = \Sigma(n, k)$ respectively. In their paper introducing forbidden poset problems, Katona and Tarján [51] considered the so-called V poset defined on $\{x, y, z\}$ with relations $x \le y, z$. They proved

$$\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \le \mathrm{La}(n, V) \le \left(1 + \frac{2}{n}\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This result is an example of an obstruction that often comes up in forbidden poset problems. The family yielding the lower bound is formed by taking a full level and a collection of sets on the next level with pairwise symmetric difference larger than two (see [34]). However, determining the optimal such family is a difficult open problem in coding theory.

Define the Λ poset the same way as V but with all the relations reversed. In the same paper, Katona and Tarján determined the following exact result for $n \ge 3$:

$$\mathrm{La}(n, V, \Lambda) = \mathrm{La}^{\#}(n, V, \Lambda) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

The extremal configuration is given by taking every set in $\binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor}$ as well as every such set union $\{n\}$.

A variety of other posets were investigated with similar extremal behavior to the V. These include batons $P_k(r, s)$ [76, 40] defined by the relations $x_1, \ldots, x_r \le y_1 \le y_2 \le \ldots \le y_{k-2} \le z_1, z_2, \ldots, z_s$, the special case V_r [23] defined by the relations $x \le y_1, y_2, \ldots, y_r$, the N poset [36] ($x, y \le w$ and $y \le z$) as well as the induced V case [15]. A far reaching result was obtained by Bukh [13] who determined the correct asymptotic bound for any poset whose Hasse diagram is a tree. In particular, Bukh showed

$$\mathrm{La}(n, T) \le (h(T) - 1)\binom{n}{\lfloor n/2 \rfloor}\left(1 + \mathcal{O}(\frac{1}{n})\right).$$

The case when $h(T) = 2$ was also settled independently by Griggs and Lu [40]. In the induced case, the asymptotic result for trees was obtained by Boehnlein and Jiang [10].

Other posets for which $\mathrm{La}(n, P)$ has been studied include harps [39], generalized diamonds [39], fans [38], crowns [58, 40] and recently the complete 3 level poset $K_{r,s,t}$ [71] among many others.

The most investigated poset whose asymptotic has yet to be determined is the diamond D_2 (for example, [6, 39]). This poset is defined by four elements $\{w, x, y, z\}$ with the relations $w \leq x, y \leq z$. The best known upper bound is due to Kramer, Martin and Young [54] who proved a bound of $(2.25 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$, the best possible bound using the Lubell function. It is conjectured that $\mathrm{La}(n, D_2)$ is asymptotically $2\binom{n}{\lfloor \frac{n}{2} \rfloor}$. It was shown by Czabarka, Dutle, Johnston and Székely [21] that there are, in fact, many families of size larger than $\Sigma(n, 2)$ so the asymptotic aspect of the conjecture is required. Better bounds were obtained in the case when the family is restricted to 3 levels including a bound of 2.208 by Axenovich, Manske and Martin [61], 2.1547 by Manske and Shen [61] and 2.15121 by Balogh, Hu, Lidický and Liu [7]. In the induced version of the D_2-free problem an upper bound of 2.58 is known [59].

Other variations of the extremal poset problem have been considered including supersaturation [70], which studies how many of a given poset we have once we pass the extremal threshold; poset packing problems [26, 50], which consider how many incomparable copies of a poset we can pack in $2^{[n]}$; problems in the linear lattice [72]; and the problem of determining the smallest maximal P-free families [33, 69].

The following result of De Bonis, Katona and Swanepoel [24] is of central importance to this thesis. Define the butterfly poset B on the ground set $\{w, x, y, z\}$ by the relations $w, x \leq y, z$.

Theorem 11 (De Bonis, Katona and Swanepoel [24])**.**

$$\mathrm{La}(n, B) = \Sigma(n, 2).$$

Moreover, equality holds if and only if the family is the union of two levels of maximal size.

Their proof used Katona's method of cyclic permutations. Two other very simple proofs of this result have been given. We will make use of techniques from both of these proofs so we will recall them both. We begin with the proof of Burcsi and Nagy.

Proof (Burcsi and Nagy [14]). First, we must introduce the notion of a *double chain*. Begin with a maximal chain

$$\mathcal{C} = \{\varnothing, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \ldots, [n]\}$$

formed by adding sequentially the elements $x_1, x_2, \ldots, x_n$. Now, to form the double chain $\mathcal{D}$ we add the additional sets $\{x_2\}, \{x_1, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_3, x_5\}, \ldots$ formed by adding each element one step earlier than we did in $\mathcal{C}$. Now, $\mathcal{D}$ contains two sets of size 1 through $n-1$ plus $\varnothing$ and $[n]$. It is easy to argue that a maximal B-free family $\mathcal{F}$ does not contain $\varnothing$ or $[n]$ ($n \geq 4$). It can be checked that exactly $2\,|F|\,!\,(n-|F|)!$ double chains contain a set of size between 1 and $n-1$. A simple way to understand this is outlined in Chapter 2 in the proof of Lemma 2. It follows that the number of pairs $(F, \mathcal{D})$ where $F \in \mathcal{F} \cap \mathcal{D}$ is

$$\sum_{F \in \mathcal{F}} 2\,|F|\,!\,(n-|F|)!\,.$$

Finally, it is also not hard to see that there are exactly $n!$ double chains, and a simple analysis shows that a double chain can have at most 4 sets from $\mathcal{F}$. Thus, the number of pairs is

$$\sum_{\mathcal{D}} 4 = 4n!\,.$$

Dividing through yields

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 2. \tag{1.5}$$

To determine the maximal B-free family from (1.5) we can invoke Lemma 1. $\square$

Next we will introduce the partition method of Griggs, Li and Lu [39, 37]. For simplicity we will introduce some notation for the Lubell function. Define, for any $\mathcal{F} \subseteq 2^{[n]}$,

$$\ell_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

One interpretation of $\ell_n(\mathcal{F})$ is that it is the average number of intersections of a random maximal chain $\mathcal{C}$ with the family $\mathcal{F}$. Let $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_r$ be a partition of all maximal chains in $2^{[n]}$ into r classes. This partition may depend on the family $\mathcal{F}$ itself. Recall that by double counting pairs

$(F, \mathcal{C})$ we have

$$\sum_{F \in \mathcal{F}} |F|\,!\,(n - |F|)! = \sum_{\mathcal{C}} |\mathcal{F} \cap \mathcal{C}|\,.$$

It follows that

$$\begin{aligned} n!\,\ell_n(\mathcal{F}) &= \sum_{i=1}^{r} \sum_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}| \\ &= \sum_{i=1}^{r} |\mathcal{P}_i| \frac{\sum_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}|}{|\mathcal{P}_i|} \\ &= \sum_{i=1}^{r} |\mathcal{P}_i| \operatorname{ave}_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}|\,, \end{aligned}$$

where $\operatorname{ave}_{\mathcal{C} \in \mathcal{P}_i}$ denotes the average over $\mathcal{C} \in \mathcal{P}_i$. Taking the maximum over $\operatorname{ave}_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}|$ we have that

$$\begin{aligned} n!\,\ell_n(\mathcal{F}) &\leq \sum_{i=1}^{r} |\mathcal{P}_i| \max_i \operatorname{ave}_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}| \\ &= n! \max_i \operatorname{ave}_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}| \end{aligned}$$

and so

$$\ell_n(\mathcal{F}) \leq \max_i \operatorname{ave}_{\mathcal{C} \in \mathcal{P}_i} |\mathcal{F} \cap \mathcal{C}|\,.$$

Thus, the goal of the method of Griggs, Li and Lu is to bound the worst-case average intersection of chains with the family across all parts of the partition. From here a proof of the upper bound on butterfly-free families is almost immediate.

Proof (Griggs, Li, Lu [39, 37]). Let $\mathcal{F}$ be a B-free family (we may assume $\varnothing, [n] \notin \mathcal{F}$). Note that, in particular, if $\mathcal{F}$ is B-free, then $\mathcal{F}$ is also Y-free ($w \leq x \leq y, z$) and Y'-free ($y, z \leq x \leq w$) since B is a subposet of Y and Y'. Let $\mathcal{M}$ be the collection of those sets in $M \in \mathcal{F}$ such that there exist $A, B \in \mathcal{F}$ with $A \subset M \subset B$. Observe that, in this case, A and B are unique in the sense that there can be no other sets containing or contained in M. Now, if a chain does not contain some $M \in \mathcal{M}$, then it contains at most 2 elements of $\mathcal{F}$. Thus, we group together all chains which don't contain a set from $\mathcal{M}$ into one class $\mathcal{P}_0$. From here, the simplest proof, noted in [39], is that each chain through A, M and B may be paired off with a chain which is otherwise identical but misses A and B. One can easily show that this is a bijection using the fact that A and B must be unique in a butterfly-free family. Thus, we may form partitions of size 2 with 3 intersections on one chain and 1 on the other, so the desired bound of 2 on the average holds. Alternatively, for each $M \in \mathcal{M}$ we may form a partition $\mathcal{P}_M$ consisting of all chains passing through M. Now, the fraction of chains passing through M which also pass through A is $\frac{1}{\binom{|M|}{|A|}}$ and the fraction that pass through B is $\frac{1}{\binom{n-|M|}{|B|}}$. Thus, we have

$$\begin{aligned}
\mathrm{ave}_{\mathcal{C}\in\mathcal{P}_M} &= 1 + \frac{1}{\binom{|M|}{|A|}} + \frac{1}{\binom{n-|M|}{|B|}} \\
&\leq 1 + \frac{1}{2} + \frac{1}{2} \\
&= 2.
\end{aligned}$$

□

The partition method generalizes an earlier technique of Katona [49] who studied partitions of the chains defined by connected components of the comparability graphs on P-free families. These methods have been among the most powerful in the determination of extremal poset problems. In Chapter 4 we will introduce two additional generalizations to the method. Namely, we will use weight functions and work with cyclic permutations instead of chains.

We conclude this chapter by discussing general bounds in terms of $h(P)$ and $|P|$. Trivially, one has $\mathrm{La}(n, P) \leq (|P| - 1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$ because every poset P is contained (in a noninduced sense) in a chain of length $|P|$, so we may apply Erdős's result on k-Sperner families. The first nontrivial result was given by Burcsi and Nagy [14] using the double chain method outlined above.

Theorem 12 (Burcsi, Nagy [14]). *For any poset P, when n is sufficiently large, we have*

$$\mathrm{La}(n, P) \leq \left(\frac{|P| + h(P)}{2} - 1\right)\binom{n}{\lfloor n/2 \rfloor}. \tag{1.6}$$

This result was improved by Chen and Li [16]. The idea of their proof was to generalize the double chain to a more complicated structure.

Theorem 13 (Chen, Li [16]). *For any poset P, when n is sufficiently large, the inequality*

$$La(n, P) \leq \frac{1}{m+1}\left(|P| + \frac{1}{2}(m^2 + 3m - 2)(h(P) - 1) - 1\right)\binom{n}{\lfloor n/2 \rfloor} \tag{1.7}$$

holds for any fixed integer $m \geq 1$.

Putting $m = \left\lceil \sqrt{\frac{|P|}{h(P)}} \right\rceil$ in the above formula, they obtained

$$\mathrm{La}(n, P) = \mathcal{O}(|P|^{1/2}\, h(P)^{1/2}) \binom{n}{\lfloor n/2 \rfloor}. \tag{1.8}$$

The main result of the next chapter is a sharpening of the result of Chen and Li, Theorem 13, and the asymptotic bound (1.8).

Chapter 2

An improvement on the general bound on the largest family of subsets avoiding a subposet

2.1 Introduction

We further improve Theorem 13, by showing that

Theorem 14 (Grósz, Methuku, T [41]). *For any poset P, when n is sufficiently large, the inequality*

$$\mathrm{La}(n,P) \le \frac{1}{2^{k-1}}\left(|P| + (3k-5)2^{k-2}(h(P)-1) - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds for any fixed $k \ge 2$.

Notice that putting $k=2$, we get Theorem 12 and Theorem 13 for $m=1$. Putting $k=3$, we get Theorem 13 for $m=3$. For $k>3$, our result strictly improves Theorem 13.

By choosing k appropriately in our theorem, we obtain the following improvement of (1.8):

Corollary 1 (Grósz, Methuku, T [41]). *For every poset P and sufficiently large n,*

$$\mathrm{La}(n,P) = \mathcal{O}\left(h(P)\log_2\left(\frac{|P|}{h(P)} + 2\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The following proposition shows that this bound cannot be improved for general P.

Proposition 1 (Grósz, Methuku, T [41]). *Let $P = K_{a,a,\dots,a}$ be the the poset defined antichains $A_1, \dots A_r$ of size a so that if $x \in A_i$ and $y \in A_j$ with $i < j$ we have $x < y$. Then,*

$$\mathrm{La}(n,P) \geq ((h(P)-2)\log_2 a)\binom{n}{\lfloor \frac{n}{2} \rfloor} = \left((h(P)-2)\log_2\left(\frac{|P|}{h(P)}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

It is interesting to note that much less is known about the induced version. The only known general bound on $\mathrm{La}^{\#}(n,P)$ has a much weaker constant than for the non-induced problem due to its dependence on the constant term of the higher dimensional variant of the Marcus-Tardos theorem [62, 52].

Theorem 15 (Methuku, Pálvölgyi [65]). *For every poset P, there is a constant C such that the size of any family of subsets of $[n]$ that does not contain an induced copy of P is at most $C\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Recall that the Lubell function of a family of subsets of $[n]$ is defined as $l_n(\mathcal{A}) = \sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}}$. The Lubell function is the sum of the proportion of sets selected of each size; clearly $l_n(\mathcal{A}) \geq \frac{|\mathcal{A}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$. Define $\lambda_n^{\#}(P)$ as the maximum value of $l_n(\mathcal{A})$ over all induced P-free families $\mathcal{A} \subset 2^{[n]}$. While $\frac{\mathrm{La}^{\#}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ is known to have a constant bound for every P, it is not currently known if $\lambda_n^{\#}(P)$ also has a constant bound for every P. We prove the following result about $\lambda_n^{\#}(P)$.

Theorem 16 (Grósz, Methuku, T [41]). *For every poset P and every $c > \frac{1}{2}$,*

$$\lambda_n^{\#}(P) = \mathcal{O}(n^c).$$

The chapter is organized as follows: In the second subsection we define our more general chain structure called an *interval chain* and give a proof of Theorem 14 and Corollary 1 using it. In the third subsection we give another proof of Corollary 1, with a better constant, using an embedding of arbitrary posets into a product of generalized diamonds. We also give a proof of Proposition 1. In the fourth subsection we use the interval chain technique to prove Theorem 22.

2.2 Interval chains and the proof of Theorem 14

We begin by proving some lemmas which allow us to extend Lubell's argument to more general structures. Let $\pi \in S_n$ be a permutation and $A \subset [n]$ be a set, then A^π denotes the set $\{\pi(a) : a \in$

$A\}$. Moreover, for a collection of sets $\mathcal{H} \subset 2^{[n]}$ we define $\mathcal{H}^\pi$ to be the collection $\{A^\pi : A \in \mathcal{H}\}$.

Lemma 2. *Let $\mathcal{H} \subset 2^{[n]}$ be a collection of sets and $A \subset [n]$ be any set. Let $N_i = N_i(\mathcal{H})$ be the number of sets in $\mathcal{H}$ of cardinality i. The number of permutations π such that $A \in \mathcal{H}^\pi$ is $N_{|A|}\,|A|\,!\,(n-|A|)!$.*

Proof. Let $S_1, \ldots, S_{N_{|A|}}$ be the collection of sets in $\mathcal{H}$ of size $|A|$. The number of permutations π such that S_i is mapped to A is $|A|\,!\,(n-|A|)!$, since we can map the elements of S_i to A arbitrarily and the elements of $[n] \setminus S_i$ to $[n] \setminus A$ arbitrarily. Moreover, no permutation π maps two sets, S_i, S_j, to A, for then $S_i^\pi = S_j^\pi$, that is $\{\pi(s) : s \in S_i\} = \{\pi(s) : s \in S_j\}$ and so $S_i = S_j$, a contradiction. Since there are $N_{|A|}$ sets in $\mathcal{H}$ of size $|A|$, and we have shown that the set of permutations mapping each of them to A is disjoint. It follows that the number of permutations π such that $A \in \mathcal{H}^\pi$ is $N_{|A|}\,|A|\,!\,(n-|A|)!$. □

For a collection $\mathcal{H} \subset 2^{[n]}$ and a poset, P, let $\alpha(\mathcal{H}, P)$ denote the size of the largest subcollection of $\mathcal{H}$ containing no P. Observe that $\alpha(\mathcal{H}, P) = \alpha(\mathcal{H}^\pi, P)$ for all $\pi \in S_n$ since containment relations are unchanged by permutations of $[n]$.

Lemma 3. *Let $\mathcal{A}$ be a P-free family in $2^{[n]}$ and $\mathcal{H}$ be a fixed collection. We have*

$$\sum_{A \in \mathcal{A}} \frac{N_{|A|}}{\binom{n}{|A|}} \leq \alpha(\mathcal{H}, P).$$

In particular, if all of the N_i are equal to the same number N, we have

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq \frac{\alpha(\mathcal{H}, P)}{N}.$$

Proof. We will double count pairs (A, π) where $A \in \mathcal{H}^\pi$. First fix a set A, then Lemma 2 shows there are $N_{|A|}\,|A|\,!\,(n-|A|)!$ permutations for which $A \in \mathcal{H}^\pi$. Now fix a permutation $\pi \in S_n$. By the definition of $\alpha(\mathcal{H}, P)$ we have $|\mathcal{A} \cap \mathcal{H}^\pi| \leq \alpha(\mathcal{H}, P)$. Since there are $n!$ permutations, it follows that the number of pairs (A, π) is at most $\alpha(\mathcal{H}, P)n!$. Thus, we have

$$\sum_{A \in \mathcal{A}} N_{|A|}\,|A|\,!\,(n-|A|)! \leq \alpha(\mathcal{H}, P)n!\,,$$

and rearranging yields the result. □

We introduce a structure $\mathcal{H} \subset 2^{[n]}$ which we call a *k-interval chain*. Define the interval $[A, B]$ to be the set $\{C : A \subseteq C \subseteq B\}$. Fix a maximal chain $\mathcal{C} = \{A_0 = \emptyset, A_1, \ldots, A_{n-1}, A_n = [n]\}$ where $A_i \subset A_{i+1}$ for $0 \leq i \leq n-1$. From $\mathcal{C}$ we define the k-interval chain $\mathcal{C}_k$ as

$$\mathcal{C}_k = \bigcup_{i=0}^{n-k} [A_i, A_{i+k}].$$

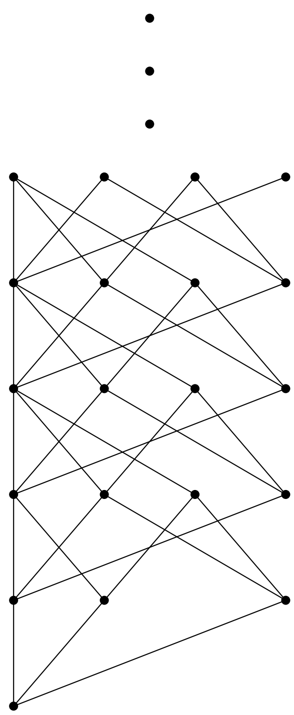

Figure 2.1: 3-interval chain

See Figure 2.1 for an example of an interval chain. We begin by deriving some properties of interval chains. In the rest of the paper we shall work with the k-interval chain $\mathcal{C}_k^0$ defined by $A_i = [i]$; other k-interval chains are related to it by permutation. It is easy to see that the indicator vectors of the sets in $\mathcal{C}_k^0$ consist of an initial segment of 1's, then k arbitrary bits, followed by 0's. We call the number of 1's in a 0–1 vector the weight of the vector (which is the size of the corresponding set).

We will now prove a sequence of lemmas that we use to bound the number of sets in a P-free subfamily of a k-interval chain. We call two sets related if one of them contains the other. The idea, following Burcsi, Nagy [14] and Chen, Li [16], is to partition P into $h(P)$ antichains and embed the antichains into a given subcollection of $\mathcal{C}_k^0$, one by one, in such a way that every set in one antichain is related to every set in the next antichain. To this end, we ignore those sets in $\mathcal{C}_k^0$ which may be unrelated to some previously embedded set. The key lemma, Lemma 5, gives an upper bound to how many sets we must ignore.

For convenience, from now on we identify sets and their indicator vectors.

Lemma 4. *For $k \le m \le n-k$, the number of sets of size m in a k-interval chain is 2^{k-1}. The number of such sets which have at least j 0's before the last 1 is $\sum_{h=j}^{k-1}\binom{k-1}{h}$.*

Proof. We give a bijection φ between 0–1 vectors of length $k-1$ and sets of size m in $\mathcal{C}_k^0$. Let u be a 0–1 vector of length $k-1$, and let w be the weight of u. Let $\varphi(u) = \overbrace{111\dots1}^{m-w-1}\overbrace{\ u\ }^{k-1}1\overbrace{0000000\dots0}^{n-m-k+w+1}$. A set of size m in $\mathcal{C}_k^0$ is assigned to u if and only if in its indicator vector the last $k-1$ bits leading up to (but not including) the last 1 coincide with u. We show φ is injective and surjective. If $\varphi(u)=\varphi(v)$, then both u and v consist of the $k-1$ bits preceding the final 1 so $u=v$, and it follows φ is injective. Now, take an arbitrary weight m vector, x, corresponding to a set in $\mathcal{C}_k^0$. Find the last 1 occurring in x and let u be the vector of length $k-1$ immediately preceding it (such a vector exists since $m \ge k$). Then $\varphi(u)=x$, and we have that φ is surjective.

There are 2^{k-1} vectors u of length $k-1$. Among such vectors, $\sum_{h=j}^{k-1}\binom{k-1}{h}$ of them have at least j 0's, and precisely these vectors are the ones mapped to vectors with at least j 0's before the last 1. The condition $k \le m \le n-k$ guarantees that both $m-w-1$ and $m+k-w+1$ are between 0 and n. □

Lemma 5. *For $3k-3 \le m \le n-k+1$, the number of sets in a k-interval chain which have size at most $m-1$, and which are unrelated to some other set in the k-interval chain of size at least m, is $(3k-5)2^{k-2}$.*

Proof. We will show that the sets in the k-interval chain $\mathcal{C}_k^0$, which are unrelated to at least one set of size m or greater in $\mathcal{C}_k^0$ are: all indicator vectors in $\mathcal{C}_k^0$ of weight between $m-1$ and $m-(k-2)$ inclusive; plus, among indicator vectors with weight $m-i$ with $k-1 \le i \le 2k-3$, those which have at least $i-k+2$ 0's before the last 1. Let's denote the collection of these vectors by $\mathcal{S}$. Then, by Lemma 4, we can calculate the number $|\mathcal{S}|$ of such vectors:

$$(k-2)2^{k-1} + \sum_{i=k-1}^{2k-3} \sum_{h=i-k+2}^{k-1} \binom{k-1}{h} = (k-2)2^{k-1} + \sum_{j=1}^{k-1}\sum_{h=j}^{k-1} \binom{k-1}{h} =$$

$$= (k-2)2^{k-1} + \sum_{h=1}^{k-1} h\binom{k-1}{h} = (k-2)2^{k-1} + (k-1)2^{k-2} = (3k-5)2^{k-2}.$$

First we show that if $v \in \mathcal{S}$, there is a vector of weight m in $\mathcal{C}_k^0$ which is unrelated to it. Let $m-i$ be the weight of v. We need to change at least one 1 to 0 (i.e., remove some elements), and change i more 0's to 1's than we just removed (that is, add i more elements than we just removed).

Assume that the last 1 in v is at index l, so the first $l-k$ elements in v are 1's. Also assume that there are j 0's in v with an index less than l. We can change v_l, the l^{th} entry of v, from 1 to 0, and change the first $i+1$ 0's in v to 1's because $i+1 \le j+k-1$. We obtain either a vector with at least $l-k+2$ initial 1's, and 0's from an index $\le l$; or a vector with $l-1$ initial 1's, and 0's from an index $\le l+k-1$ (see the figure below). Either way the difference between the index of the last 1 and the first 0 is at most $k-1$, so the obtained vector is in $\mathcal{C}_k^0$.

$$\overbrace{111111111}^{\text{initial segment}}\ \overbrace{00010}^{\le k-1}1\overbrace{00000}^{k-1}000 \qquad\qquad \overbrace{111111111}^{\text{initial segment}}\ \overbrace{00010}^{\le k-1}1\overbrace{00000}^{k-1}000$$

$$\downarrow \qquad\qquad \text{or} \qquad\qquad \downarrow$$

$$\overbrace{111111111}\overbrace{11010}^{\le k-1}0\overbrace{00000}^{k-1}000 \qquad\qquad \overbrace{111111111}\overbrace{11111}^{\le k-1}0\overbrace{11000}^{k-1}000$$

Conversely, we prove that if v (which is of weight at most $m-i$, $i \ge 1$) is not in $\mathcal{S}$, then it is related to all vectors of weight at least m in $\mathcal{C}_k^0$. Assume by contradiction that it is unrelated to a vector q in $\mathcal{C}_k^0$, of weight at least m.

Consider the transformation of v into q by changing some 1's to 0's and some 0's to 1's. Let l' be the index of the first 1 that we change to 0. Then $l' \le l$ (in the transformation given above, it was l, the index of the last 1). We can only change those bits from 0's to 1's which are before l' (at most j), or those which are between $l'+1$ and $l'+k-1$ (at most $k-1$); this is because the new vector will have a 0 at index l' and so it cannot have 1's after index $l'+k-1$ if it is in $\mathcal{C}_k^0$. So if $i+1 > j+k-1$, there are not enough 0's which could be changed to 1's, so we cannot obtain a vector of weight m or greater, which is in $\mathcal{C}_k^0$ and is unrelated to it. $\square$

Observation 1. The sets in $\mathcal{C}_k^0$ which are related to every set of size at least $m+1$ in $\mathcal{C}_k^0$, but unrelated to at least one set of size m in $\mathcal{C}_k^0$ are those which have size $m-i$ with $k-2 \le i \le 2k-3$, and in whose indicator vector the number of 0's before the last 1 is exactly $i-k+2$. The only way we can obtain an indicator vector of weight m corresponding to such a set in $\mathcal{C}_k^0$ is by removing the last 1, and changing all 0's before the last 1, plus the next $k-1$ after it, to 1's. Thus, there is only one set of size m in $\mathcal{C}_k^0$ which is unrelated to these sets: the one with an indicator vector $\overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$.

Lemma 6. *For any poset P of size $|P|$ and height h, we have*

$$\alpha(\mathcal{C}_k, P) \le |P| + (h-1)(3k-5)2^{k-2} - 1.$$

Proof. We show that if $\mathcal{H} \subseteq \mathcal{C}_k^0$ with $|\mathcal{H}| \ge |P| + (h-1)(3k-5)2^{k-2}$, then $\mathcal{H}$ contains P as a subposet. We may notice that a k-interval chain on $[n]$ is a subposet of the levels $3k-3$ to $n'-k+1$ of a k-interval chain on the larger base set $[n']$ where $(n'-k+1)-(3k-3) = n$ (i.e., $n' = n+4k-4$), with the injection $2^{[n]} \ni A \mapsto \{1, 2, \ldots, 3k-3\} \cup \{a+3k-3 : a \in A\} \in 2^{[n']}$. So we can assume that the elements of P are embedded from levels $3k-3$ to $n-k+1$ of the interval chain.

We define an order on $\mathcal{H}$: bigger sets come first; within sets of a given size m, the order is arbitrary, except if the set with the indicator vector $\overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$ is present in $\mathcal{H}$, it must come last among the sets of size m.

Mirsky's theorem [68] states that the height of any poset equals the minimum number of antichains into which it can be partitioned. We decompose P into antichains $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_h$, where the elements in $\mathcal{A}_i$ are bigger than or unrelated to elements in $\mathcal{A}_j$ for any $i > j$ and then map the antichains $\mathcal{A}_h, \mathcal{A}_{h-1}, \ldots, \mathcal{A}_1$ into $\mathcal{H}$ one after another, in this order, in h steps as follows. First, we map the elements of $\mathcal{A}_h$ to the first $|\mathcal{A}_h|$ sets of $\mathcal{H}$ in the order just described. The family of these elements of $\mathcal{H}$ is denoted $\mathcal{H}_h$. We then remove all sets in $\mathcal{H}$ which are not proper subsets of every set in $\mathcal{H}_h$. The family of these removed sets is denoted $\mathcal{I}_h$; in other words, $\mathcal{I}_h$ is the family of sets in $\mathcal{H}$ which are not properly contained in at least one set of $\mathcal{H}_h$. (Notice that $\mathcal{H}_h \subseteq \mathcal{I}_h$.) Now we map $\mathcal{A}_{h-1}$ to the first $|\mathcal{A}_{h-1}|$ sets of $\mathcal{H} \setminus \mathcal{I}_h$, denoted $\mathcal{H}_{h-1}$. We proceed similarly: we denote the family of the sets in $\mathcal{H}$ which are not properly contained in every set of $\mathcal{H}_h \cup \ldots \cup \mathcal{H}_i$ with $\mathcal{I}_i$, and map $\mathcal{A}_{i-1}$ to the collection of first $|\mathcal{A}_{i-1}|$ sets of $\mathcal{H} \setminus \mathcal{I}_i$, denoted $\mathcal{H}_{i-1}$. By this process, each set in $\mathcal{H}_i$ contains all the sets in $\mathcal{H}_j$ for $i > j$.

We have to show that the process finishes before $\mathcal{H}$ is exhausted, that is,

$$\left| \bigcup_{i=1}^{h} \mathcal{H}_i \cup \bigcup_{i=2}^{h} \mathcal{I}_i \right| \leq |P| + (h-1)(3k-5)2^{k-2}. \tag{2.1}$$

For this purpose, we show that for each $i \in \{h, h-1, \ldots, 2\}$, the number of new sets that are removed at this step, besides $\mathcal{H}_i$: $|\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})|$ is at most $(3k-5)2^{k-2}$ (where we consider $\mathcal{I}_{h+1} = \emptyset$). Since $\left|\bigcup_{i=1}^{h} \mathcal{H}_i\right| = |P|$ and there are $h(P)-1$ steps in which sets are removed, we will have our desired inequality (2.1). Let A be the last set in $\mathcal{H}_i$ in the order we defined on $\mathcal{H}$, and $m = |A|$. Every set which comes before A is either in $\mathcal{H}_i$ or $\mathcal{I}_{i+1}$. If $A = \overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$, then $\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})$ is a subcollection of all sets in $\mathcal{C}_k^0$ whose size is smaller than m, but which are unrelated to at least one set in $\mathcal{C}_k^0$ of size m or more. By Lemma 5, the number of such sets is $(3k-5)2^{k-2}$. If $A \neq \overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$, then, by Observation 1, the sets in $\mathcal{C}_k^0$ whose size is smaller than m, and which are unrelated to A or some other set in $\mathcal{H}$ which is smaller than A in our order, are also unrelated to some set in $\mathcal{C}_k^0$ of size $m+1$ or more. Thus the sets in $\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})$ are some sets in $\mathcal{C}_k^0$ of size m and some sets whose size is smaller than m but unrelated to at least one set in $\mathcal{C}_k^0$ of size $m+1$ or more. Again, the number of such sets is at most $(3k-5)2^{k-2}$. $\square$

Now we are ready to prove our main result, Theorem 14.

Proof of Theorem 14. Let $\mathcal{A}$ be a P-free family over $[n]$. Let $N_{|A|}$ denote the number of sets of size $|A|$ from the k-Interval chain.

$$\begin{aligned} 2^{k-1}|\mathcal{A}| &= \sum_{\substack{A\in\mathcal{A} \\ |A|<k \text{ or } |A|>n-k}} 2^{k-1} + \sum_{\substack{A\in\mathcal{A} \\ k\leq|A|\leq n-k}} 2^{k-1} \\ &\leq \sum_{\substack{A\in\mathcal{A} \\ |A|<k \text{ or } |A|>n-k}} \frac{N_{|A|}\binom{n}{\lfloor \frac{n}{2}\rfloor}}{\binom{n}{|A|}} + \sum_{\substack{A\in\mathcal{A} \\ k\leq|A|\leq n-k}} \frac{2^{k-1}\cdot\binom{n}{\lfloor \frac{n}{2}\rfloor}}{\binom{n}{|A|}} \leq \alpha(\mathcal{C}_k, P)\binom{n}{\lfloor \frac{n}{2}\rfloor}. \end{aligned}$$

If $|A| < k$ or $|A| > n-k$, we have $2^{k-1} \leq \frac{\binom{n}{\lfloor \frac{n}{2}\rfloor}}{\binom{n}{|A|}}$ when n is sufficiently large and so the first inequality holds. If $k \leq |A| \leq n-k$, by Lemma 4, we have $2^{k-1} = N_{|A|}$ and so the second inequality holds due to Lemma 3. Now we use Lemma 6 to upper bound $\alpha(\mathcal{C}_k, P)$, from which the theorem follows. $\square$

We now obtain Corollary 1 using the above theorem.

First proof of Corollary 1. Let $\mathcal{A}$ be a P-free family, and let h be the height of P. Define $k = \left\lceil \log_2\left(\frac{|P|}{h}\right)\right\rceil = \log_2\left(\frac{|P|}{h}\right) + x = \log_2\left(\frac{|P|y}{h}\right)$. Let us substitute this k into Theorem 14 (where $0 \leq x < 1$ and $1 \leq y < 2$). If $k \geq 2$, we get

$$\frac{|\mathcal{A}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \frac{1}{2^{k-1}}\left(|P| + (h-1)(3k-5)2^{k-2} - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} < \frac{3 \cdot 2^{k-2}kh + |P|}{2^{k-1}} =$$

$$= \frac{\frac{3}{4}y\,|P|\left(\log_2\left(\frac{|P|}{h}\right) + x\right) + |P|}{\frac{y|P|}{2h}} < \frac{3}{2}\log_2\left(\frac{|P|}{h}\right)h + 3.5h.$$

If $k \leq 1$, we have $|P| \leq 2h$. Double counting with just the chain gives a bound of $|P|\binom{n}{\lfloor \frac{n}{2} \rfloor}$ (see Erdős [28]), so the corollary still holds. So we have,

$$\mathrm{La}(n, P) < \left(\frac{3}{2}\log_2\left(\frac{|P|}{h}\right)h + 3.5h\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad \square$$

2.3 A different proof of Corollary 1 using generalized diamonds

We begin by recalling some results from the papers of Griggs and Li [38] and Griggs, Li and Lu [39].

Definition 1 (Product of posets)**.** *If a poset P has a unique maximal element and a poset Q has a unique minimal element, then their product $P \otimes Q$ is defined as the poset formed by identifying the maximal element of P with the minimal element of Q.*

Lemma 7 (Griggs, Li [38])**.** $\mathrm{La}(n, P \otimes Q) \leq \mathrm{La}(n, P) + \mathrm{La}(n, Q)$.

Proof. Let $\mathcal{F}$ be a maximal $P \otimes Q$-free family. Define $\mathcal{F}_1 = \{S \in \mathcal{F} \mid \mathcal{F} \cap [S, [n]] \text{ contains } Q\}$ and let $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$.

We claim that $\mathcal{F}_1$ is P-free. Suppose not. Then there is a set $M_1 \in \mathcal{F}_1$ which represents the maximal element of P, and, by definition, $\mathcal{F} \cap [M_1, [n]]$ contains Q. Also notice that, since M_1

represents the maximal element of P, there are no elements in $[M_1, [n]] \setminus \{M_1\}$ that are part of the representation of P. This implies that $\mathcal{F}$ contains $P \otimes Q$, a contradiction. It is easy to see that $\mathcal{F}_2$ is Q-free, for otherwise, the element M_2, that represents the minimal element of Q satisfies: $\mathcal{F} \cap [M_2, [n]]$ contains Q, contradicting the definition of $\mathcal{F}_2$. So we have $|\mathcal{F}| = \mathrm{La}(n, P \otimes Q) = |\mathcal{F}_1| + |\mathcal{F}_2| \le \mathrm{La}(n, P) + \mathrm{La}(n, Q)$, as desired. □

We shall write h in place of $h(P)$ for convenience. Let D_k be the poset on $k+2$ elements with relations $b < c_1, c_2, \ldots, c_k < d$. Let $K_{a_1,\ldots,a_h}$ be the complete h-level poset where the sizes of levels are $a_1, a_2, \ldots, a_h$: the poset in which every element is smaller than every element on every higher level.

By using a partition method on chains, Griggs, Li and Lu proved

Theorem 17 (Griggs, Li, Lu [39]). *Let $k \ge 2$. Then,*

$$\mathrm{La}(n, D_k) \le (\log_2(k+2) + 2)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

By Mirsky's decomposition [68], P can be viewed as a union of h antichains: $\mathcal{A}_i$, $1 \le i \le h$. Let $|\mathcal{A}_i| = a_i$. Then, it is easy to see that the following lemma holds.

Lemma 8. *P is a subposet of $K_{a_1,\ldots,a_h}$, which in turn, is a subposet of $D_{a_1} \otimes D_{a_2} \otimes \ldots \otimes D_{h-1} \otimes D_{a_h}$.*

Now we are ready to prove Corollary 1 with better constants.

Second proof of Corollary 1. By Lemma 8, we have

$$\mathrm{La}(n, P) \le \mathrm{La}(n, K_{a_1,\ldots,a_h}) \le \mathrm{La}(n, D_{a_1} \otimes D_{a_2} \otimes \ldots \otimes D_{a_{h-1}} \otimes D_{a_h}).$$

By Lemma 7 and Theorem 17, we have

$$\mathrm{La}(n, D_{a_1} \otimes D_{a_2} \otimes \ldots \otimes D_{a_{h-1}} \otimes D_{a_h}) \le \sum_{i=1}^{h} (\log_2(a_i + 2) + 2)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Bounding the sum on the right-hand side, by Jensen's inequality we have

$$\sum_{i=1}^{h}(\log_2(a_i+2)+2) \leq h \cdot \log_2\left(\frac{|P|}{h}+2\right)+2h.$$

This implies our desired result

$$\mathrm{La}(n,P) \leq \left(h \cdot \log_2\left(\frac{|P|}{h}+2\right)+2h\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad \square$$

Finally, we will prove Proposition 1, a matching lower bound for Corollary 1.

Proof of Proposition 1. We show that the height of any poset corresponding to a family of sets which realizes $K_{a,a,\dots,a}$ is at least $(h-2)\log_2 a+1$. This implies that if $\mathcal{A}$ is the middle $(h-2)\log_2 a$ levels of $2^{[n]}$, it does not contain P as a subposet.

Let us denote the levels of $P = K_{a,a,\dots,a}$ by $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_h$, and let $\mathcal{H}$ be a set family into which P is embedded. For every $1 \leq i \leq h-1$, let U_i be the union of the sets corresponding to the elements of $\mathcal{P}_i$ by the embedding. Then, the structure of P implies that every element of $\mathcal{P}_{i+1}$ is mapped to sets containing U_i. If $|U_{i+1} \setminus U_i| = k$, there are 2^k sets in total containing U_i and contained in U_{i+1}. Thus, we have $|U_{i+1}| - |U_i| \geq \log_2 a$ (this idea comes from Theorem 2.5 in [39]). So $|U_{h-1}| - |U_1| \geq (h-2)\log_2 a$. $\mathcal{P}_1$ is mapped to sets of size at most $|U_1|$, and $\mathcal{P}_h$ is mapped to sets of size at least $|U_{h-1}|$, so the set family spans at least $(h-2)\log_2 a + 1$ levels. $\square$

2.4 Proof of Theorem 16

In this subsection we will give an upper bound on the size of the Lubell function of an induced P-free family. Lemma 3 holds for induced posets as well by an identical proof. Let $0 \leq a \leq b \leq n$. Let $\mathcal{H} \subset 2^{[n]}$ be a collection of sets which has the same number of sets, N, for each cardinality i for $a \leq i \leq b$. Define $\alpha^{\#}(\mathcal{H}, P)$ to be the size of the largest subcollection of $\mathcal{H}$ containing no induced P.

Lemma 9. *Let $\mathcal{A}$ be an induced P-free family in $2^{[n]}$, in which the cardinality of every set is between a and b. We have*

$$l_n(\mathcal{A}) \leq \frac{\alpha^{\#}(\mathcal{H}, P)}{N}.$$

In particular, if $\mathcal{C}_k$ is an interval chain as defined in the Section 2.2, and $k \le a$ and $b \le n-k$ hold, we have

$$l_n(\mathcal{A}) \le \frac{\alpha^{\#}(\{A \in \mathcal{C}_k : a \le |A| \le b\}, P)}{2^{k-1}}.$$

Proof. The proof of Lemma 3 applies, observing that $a \le |A| \le b$. □

We prove the following statement, which is slightly stronger than Theorem 22.

Lemma 10. *Let P be a poset and let $c > \frac{1}{2}$. Let n be a natural number, and let $0 \le a \le b \le n$. If $\mathcal{A}$ is an induced P-free family in which the cardinality of every set is between a and b,*

$$l_n(\mathcal{A}) = \mathcal{O}\left((b-a)^c\right).$$

The following claim will be used recursively and is key to the proof of our lemma.

Claim 1. *If Lemma 10 holds for a given $c = c' > \frac{1}{2}$, then it also holds for $c = \frac{2c'}{2c'+1}$.*

Proof of Claim. Let $m = b - a + 1$, and let $k = m^{\frac{2}{2c'+1}}$. Let $\mathcal{H} = \{A \in \mathcal{C}_k : a + k \le |A| \le b - k\}$. By definition $\mathcal{C}_k = \bigcup_{i=0}^{n-k} [A_i, A_{i+k}]$ (where $A_0 \subset A_1 \subset \ldots \subset A_n$ is an arbitrary maximal chain), and the levels $a+k$ to $b-k$ intersect $m-k$ of the intervals $[A_i, A_{i+k}]$. By substituting k in the place of n in Theorem 15, there is a constant C such that $|\mathcal{A} \cap [A_i, A_{i+k}]| \le C\binom{k}{\lfloor \frac{k}{2} \rfloor}$ for every i. Thus $\alpha^{\#}(\mathcal{H}, P) \le (m-k)C\binom{k}{\lfloor \frac{k}{2} \rfloor} < Cm\binom{k}{\lfloor \frac{k}{2} \rfloor}$. By Lemma 9,

$$l_n(\{A \in \mathcal{C}_k : a+k \le |A| \le b-k\}) \le Cm\frac{\binom{k}{\lfloor \frac{k}{2} \rfloor}}{2^{k-1}} \le \frac{2\sqrt{2}}{\sqrt{\pi}}C\frac{m}{\sqrt{k}} = \frac{2\sqrt{2}}{\sqrt{\pi}}C\frac{m}{\sqrt{m^{\frac{2}{2c'+1}}}} = \frac{2\sqrt{2}}{\sqrt{\pi}}Cm^{\frac{2c'}{2c'+1}}. \tag{2.2}$$

By our assumption, using Lemma 10 with substituting $a + k - 1$ in the place of b, we have

$$l_n(\{A \in \mathcal{C}_k : a \le |A| \le b-k-1\}) = \mathcal{O}\left(k^{c'}\right) = \mathcal{O}\left(m^{\frac{2c'}{2c'+1}}\right). \tag{2.3}$$

Similarly, by substituting $b - k + 1$ in the place of a, we have

$$l_n(\{A \in \mathcal{C}_k : b-k+1 \le |A| \le b\}) = \mathcal{O}\left(m^{\frac{2c'}{2c'+1}}\right). \tag{2.4}$$

Adding up the inequalities (2.2), (2.3) and (2.4), we get

$$l_n(\{A \in \mathcal{C}_k : a \leq |A| \leq b\}) = \frac{2\sqrt{2}}{\sqrt{\pi}} C m^{\frac{2c'}{2c'+1}} + 2\mathcal{O}\left(m^{\frac{2c'}{2c'+1}}\right) = \mathcal{O}\left((b-a)^{\frac{2c'}{2c'+1}}\right). \qquad \square$$

Proof of Lemma 10. The lemma is trivial for $c = 1$. Substituting $c = 1$ in the proof of the claim directly gives a proof for $c = \frac{2}{3}$. Then, applying the claim recursively proves the statement for a sequence of exponents $c = c_i = \frac{2^i}{2^{i+1}-1}$. Indeed,

$$\frac{2c_i}{2c_i + 1} = \frac{2\frac{2^i}{2^{i+1}-1}}{2\frac{2^i}{2^{i+1}-1} + 1} = \frac{2^{i+1}}{2^{i+2} - 1} = c_{i+1}.$$

The limit of the sequence is $\frac{1}{2}$, so it eventually becomes smaller than any $c > \frac{1}{2}$, proving our lemma. $\square$

Chapter 3

A cyclic chain decomposition method for forbidden subposets

3.1 Introduction

Our first new result is a strengthening of the theorem of De Bonis, Katona and Swanepoel on the butterfly poset. Namely, we introduce a poset S which contains the butterfly as a strict subposet and prove that, nonetheless, the same bound holds. This poset, which we call the "skew"-butterfly, is defined by 5 elements, a, b, c, d, e, with $a, b \leq c, d$ and $b \leq e \leq d$ (see Figure 3.1).

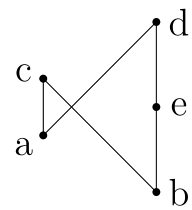

Figure 3.1: The skew-butterfly poset

Theorem 18 (Methuku, T [66])**.** *Let $n \geq 3$, then we have*

$$\mathrm{La}(n, S) = \Sigma(n, 2).$$

A construction matching this bound is given by taking two consecutive middle levels of $2^{[n]}$.

With this result (and all of the others) we also get the corresponding LYM-type inequality if we assume $\varnothing$ and $[n]$ are not in the family.

Theorem 19 (Methuku, T [66])**.** *Let $n \geq 3$ and $\mathcal{A} \subset 2^{[n]}$ be a collection of sets not containing S as a subposet, and assume that $\varnothing, [n] \notin \mathcal{A}$, then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 2.$$

For the proof of Theorem 19, we consider the set of intervals along a cyclic permutation (following Katona [45]). We partition these intervals into chains and consider the interactions of consecutive chains in the partition. The method and the proof of this result are given in Subsection 3.2.

We now mention some notable properties of S. It is one of the two posets whose Hasse diagram is a 5-cycle. The other is the harp, $H(4,3)$, and $\mathrm{La}(n, H(4,3))$ was determined exactly in the paper of Griggs, Li and Lu [39](the 4-cycles are B and D_2). The skew-butterfly is contained in the X $(a, b \leq c \leq d, e)$, a tree of height 3, like B, and so its asymptotics are determined by Bukh's theorem. The exact value of $\mathrm{La}(n, S)$ cannot be determined by the double chain method of Burcsi and Nagy [14] because one can find 5 sets on a double chain with no copy of S. Finally, if we subdivide any of the edges ac, ad or bc in the Hasse diagram of S, we get a poset for which there is a construction of size larger than $\Sigma(n, 2)$.

Next, we consider a generalization of De Bonis, Katona and Swanepoel's theorem in a different direction. If instead of forbidding B, we forbid the pair of posets Y and Y' where Y is the poset on 4 elements w, x, y, z with $w \leq x \leq y, z$ and Y' is the same poset but with all relations reversed, then $\mathrm{La}(n, Y, Y') = \mathrm{La}(n, B) = \Sigma(n, 2)$. This result is already implicit in the proof of De Bonis, Katona and Swanepoel. We extend the result by considering the posets Y_k and Y'_k defined by $k+2$ elements $x_1, x_2, \ldots, x_k, y, z$ with $x_1 \leq x_2 \leq \ldots \leq x_k \leq y, z$ and its reverse (so $Y = Y_2$ and $V = Y_1$). We prove

Theorem 20 (Methuku, T [66])**.** *Let $k \geq 2$ and $n \geq k+1$, then*

$$\mathrm{La}(n, Y_k, Y'_k) = \Sigma(n, k).$$

A construction matching this bound is given by taking k consecutive middle levels of $2^{[n]}$. We also have the LYM-type inequality:

Theorem 21 (Methuku, T [66]). *Let $k \geq 2$ and $n \geq k+1$. Assume that $\mathcal{A} \subset 2^{[n]}$ contains neither Y_k nor Y'_k as a subposet, and $\varnothing, [n] \notin \mathcal{A}$, then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq k.$$

We note that, again, the double chain method does not work for these pairs because one can have $2k+1$ sets on a double chain with no Y_k and no Y'_k by taking them consecutively on the secondary chain. We also note that, for this particular result, we can find another proof using the chain partitioning method of Griggs, Li and Lu [39] in addition to the approach described in this chapter.

Finally, we consider the more difficult induced case. We prove

Theorem 22 (Methuku, T [66]). *For $n \geq 3$, we have*

$$\mathrm{La}^{\#}(n, Y, Y') = \Sigma(2, n).$$

We also have the LYM-type inequality:

Theorem 23 (Methuku, T [66]). *Assume that $\mathcal{A} \subset 2^{[n]}$ contains neither Y nor Y' as an induced subposet, and $\varnothing, [n] \notin \mathcal{A}$, then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 2.$$

To prove Theorem 23, we introduce a second chain partitioning argument along the cycle. These partitions may be thought of as the analogue of orthogonal symmetric chain partitions for the cycle. The method and the proof of Theorem 22 are given in Subsection 3.3. This chapter is organized as follows. In the second subsection we introduce the first chain decomposition and determine $\mathrm{La}(n, S)$. In the third subsection we use the same decomposition to find $\mathrm{La}(n, Y_k, Y'_k)$ for all $k \geq 2$. In the last subsection we introduce the second decomposition and show that $\mathrm{La}^{\#}(n, Y, Y') = \Sigma(n, 2)$.

3.2 Forbidding S and the first cycle decomposition

A cyclic permutation, σ, is a cyclic ordering $x_1, x_2, \dots, x_n, x_1$ of the elements of $[n]$. We refer to the sets $\{x_i, x_{i+1}, \dots, x_{i+t}\}$, with addition taken modulo n, as intervals along the cyclic permutation. For our purpose we will not consider $\varnothing$ or $[n]$ to be intervals. The following lemma is the essential ingredient of the proof of Theorem 19:

Lemma 11. *If $\mathcal{A}$ is a collection of intervals along a cyclic permutation σ of $[n]$ which does not contain S as a subposet, then*

$$|\mathcal{A}| \leq 2n.$$

To prove Lemma 11 we will work with a decomposition of the intervals along σ into maximal chains. Set $\mathcal{C}_i = \{\{x_i\}, \{x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i+1}\}, \dots, \{x_i, x_{i-1}, \dots, x_{i+n/2-1}\}\}$ when n is even, and set $\mathcal{C}_i = \{\{x_i\}, \{x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i+1}\}, \dots, \{x_i, x_{i-1}, \dots, x_{i-(n-1)/2}\}\}$ when n is odd, where $1 \leq i \leq n$ (see Figure 3.2). Observe that the set of chains $\{\mathcal{C}_i\}_{i=1}^n$ forms a partition of the intervals along σ. We will refer to this partition as the *chain decomposition* of σ. Additionally, chains corresponding to consecutive elements of σ are called consecutive chains.

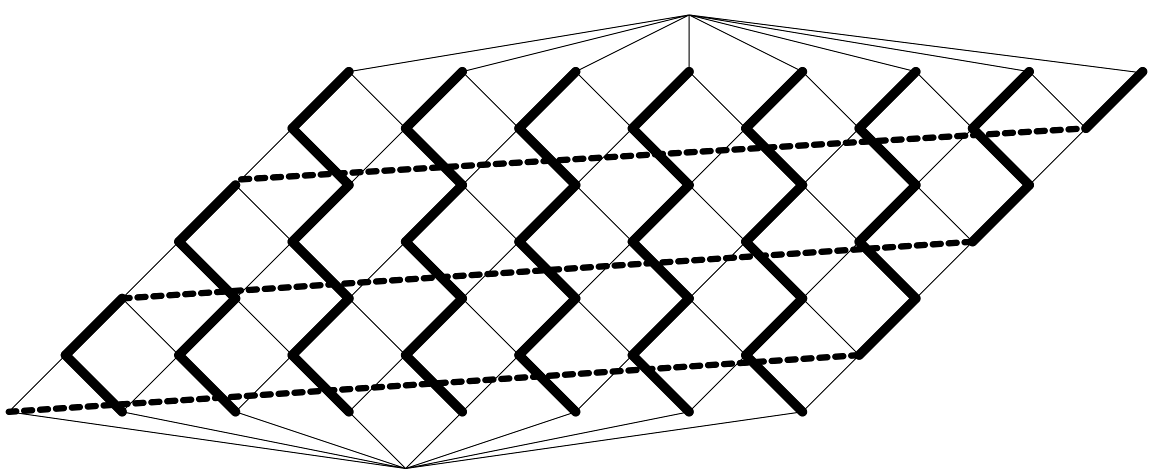

Figure 3.2: The chain decomposition is marked with bold lines on the poset of intervals along σ. The dashed lines indicate how the chains wrap around.

If $\mathcal{A}$ does not contain S as a subposet, and $\mathcal{C}$ is a chain from the chain decomposition of σ, then it is easy to see that $|\mathcal{A} \cap \mathcal{C}| \leq 4$. We will classify the chains in the chain decomposition by their intersection pattern with $\mathcal{A}$. If $|\mathcal{A} \cap \mathcal{C}| = k$, then we say $\mathcal{C}$ is of type k. When $k = 3$ we distinguish

3 cases (see Figure 3.3 for an example of each case). If $\mathcal{C}$ contains exactly 3 elements of $\mathcal{A}$, not all occurring consecutively on $\mathcal{C}$, then we say $\mathcal{C}$ is type 3^S (S for separated). If $\mathcal{C}$ has exactly 3 elements of $\mathcal{A}$ occurring consecutively with two sets of odd size, then $\mathcal{C}$ is type 3^R (facing right). If $\mathcal{C}$ has exactly 3 elements of $\mathcal{A}$ occurring consecutively with two sets of even size, then $\mathcal{C}$ is type 3^L (facing left).

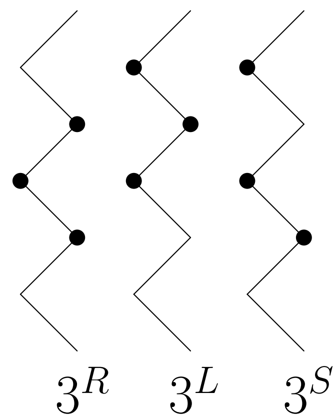

Figure 3.3: An example of chains of types 3^R, 3^L and 3^S are drawn. The elements of $\mathcal{A} \cap \mathcal{C}$ are highlighted for each type.

We will now prove a sequence of lemmas showing which types of chains can occur consecutively in the chain decomposition of σ. These lemmas will let us disregard the exact intersection pattern of $\mathcal{A}$ with the chains and allow us to work instead with the sequence of chain types.

Lemma 12. *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be two consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_i$ is of type $4, 3^R$ or 3^S, then $|\mathcal{A} \cap \mathcal{C}_{i+1}| \leq 1$.*

Proof. First, note that if $\mathcal{C}_i$ is of type 4, then we can remove a set from $\mathcal{A} \cap \mathcal{C}_i$ to make it type 3^S. Hence, we may assume that $\mathcal{C}_i$ is of type 3^S or 3^R.

In order to reduce case analysis, we will now argue that we only need to consider certain configurations of sets from $\mathcal{A}$ in $\mathcal{C}_i \cup \mathcal{C}_{i+1}$. Consider the Hasse diagram of $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ as a graph (see Figure 3.4). Call the vertices corresponding to sets in $\mathcal{A}$ occupied and the rest unoccupied. If either the top or bottom vertex in the chain is occupied, then we extend $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ in both directions maintaining the same relations between adjacent levels. Then, every occupied vertex either has degree 2 or degree 4. We will see that it is sufficient to consider the case when only degree 2 vertices are occupied. Indeed, if instead of taking a degree 4 vertex, we take an adjacent unoccupied degree

2 vertex, then no additional containments are introduced. It follows that if there was no S initially, then there will still be no S. If $\mathcal{C}_i$ is of type 3^R or 3^S, then every occupied vertex of degree 4 can be replaced by a distinct adjacent unoccupied vertex of degree 2 (This cannot be done if $\mathcal{C}_i$ is type 3^L). Thus, we may assume that all of the occupied vertices in $\mathcal{C}_i$ from the the Hasse diagram of $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ have degree 2.

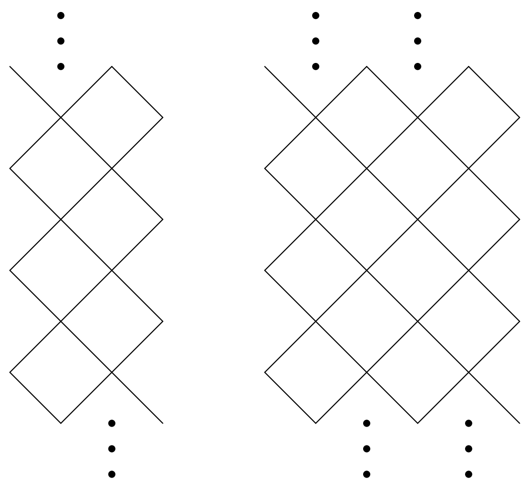

Figure 3.4: Hasse diagrams of $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ and $\mathcal{C}_i \cup \mathcal{C}_{i+1} \cup \mathcal{C}_{i+2}$ are drawn.

Let the sets in $\mathcal{A} \cap \mathcal{C}_i$ be L, M and N with $L \subset M \subset N$. Assume, by contradiction, that there are two sets $A, B \in \mathcal{A} \cap \mathcal{C}_{i+1}$ with $A \subset B$. We may assume that A and B correspond to degree 2 vertices in $\mathcal{C}_i \cap \mathcal{C}_{i+1}$. We will distinguish three cases by comparing the sizes of A and B with the size of M. If $|A| < |M| < |B|$, then L, M, N, A, B forms a skew-butterfly with $L, A \subset N, B$ and $L \subset M \subset N$. If $|M| < |A| < |B|$, then L, M, N, A, B forms a skew-butterfly with $L, M \subset N, B$ and $L \subset A \subset B$. The case $|A| < |B| < |M|$ is symmetric. It follows that there can be at most one set in $\mathcal{A} \cap \mathcal{C}_{i+1}$.

$\square$

Lemma 13. *Let $\mathcal{C}_i, \mathcal{C}_{i+1}$ and $\mathcal{C}_{i+2}$ be three consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_i$ is of type $4, 3^R$ or 3^S and $|\mathcal{A} \cap \mathcal{C}_{i+1}| = 1$, then $\mathcal{C}_{i+2}$ is of type $0, 1, 2$ or 3^R.*

Proof. By contradiction, suppose $\mathcal{C}_i$ is type $4, 3^R$ or 3^S, $|\mathcal{A} \cap \mathcal{C}_{i+1}| = 1$ and $\mathcal{C}_{i+2}$ is type $3^L, 3^S$ or 4. If $\mathcal{C}_i$ or $\mathcal{C}_{i+2}$ is of type 4, then we may disregard one set to make it type 3^S. By similar reasoning as used in Lemma 12, we may assume all occupied vertices on the Hasse diagram of $\mathcal{C}_i \cup \mathcal{C}_{i+1} \cup \mathcal{C}_{i+2}$

from $\mathcal{C}_i$ and $\mathcal{C}_{i+2}$ have degree 2. Let L, M, N be the three sets in $\mathcal{A} \cap \mathcal{C}_i$ in increasing order, and let A, B, C be the three sets in $\mathcal{A} \cap \mathcal{C}_{i+2}$ in increasing order. Without loss of generality, we may assume $|M| > |B|$. This, in turn, implies that $|M| = |B| + 1$ for otherwise L, M, N, A, B would be a skew-butterfly with $L, A \subset M, N$ and $A \subset B \subset N$. We will consider the possible locations of the set $S \in \mathcal{A} \cap \mathcal{C}_{i+1}$ on $\mathcal{C}_{i+1}$. If $|S| \leq |B|$, then N, S, A, B, C is a skew-butterfly with $A, S \subset N, C$ and $A \subset B \subset C$. If $|S| > |B|$, then L, M, N, S, A is a skew-butterfly with $L, A \subset N, S$ and $L \subset M \subset N$. Thus, in either case we have a contradiction. □

By symmetry, we also have the following corollaries of Lemmas 12 and 13:

Corollary 2. *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be two consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_{i+1}$ is of type $4, 3^L$ or 3^S, then $|\mathcal{A} \cap \mathcal{C}_i| \leq 1$.*

Corollary 3. *Let $\mathcal{C}_i, \mathcal{C}_{i+1}$ and $\mathcal{C}_{i+2}$ be three consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_{i+2}$ is of type $4, 3^L$ or 3^S and $|\mathcal{A} \cap \mathcal{C}_{i+1}| = 1$, then $\mathcal{C}_i$ is of type $0, 1, 2$ or 3^L.*

We now have sufficient information about which consecutive chain types are allowed to prove Lemma 11:

Proof of Lemma 11. We must show that the average intersection of $\mathcal{A}$ with chains from the decomposition is at most 2. To this end, we will form groups of chains such that the number of sets from $\mathcal{A}$ in each group is at most twice the size of that group.

First, consider chains of type 4. If there is a sequence of chains alternating between type 4 and type 0 spanning every chain in the chain decomposition, then it is easy to see that the average is at most 2. Otherwise, take each maximal group of consecutive chains alternating between type 0 and type 4, beginning and ending with a type 4 chain. Call such a group a 4-0-4 pattern (it may just consist of a single chain of type 4). If the group has length ℓ, then there are $2\ell + 2$ sets contributed from $\mathcal{A}$. We will add additional chains to this group to decrease the average to 2. By Lemma 12, if the chain following the type 4 chain on either side is not type 0, then it must be type 1. In this case, we add the type 1 chain to the group. Otherwise, we have a type 0 chain followed by a chain of type 0,1,2 or 3. If it is type 3, we add both the type 0 and type 3 chain to our group. Otherwise, we just add the type 0 chain. In any case, if we have added k more chains to our group (on both sides of the 4-0-4 pattern), then we have added a total of at most $2k - 2$ more sets from $\mathcal{A}$. Thus, in total, the group now consists of $k + \ell$ chains having at most $2k + 2\ell$ sets from $\mathcal{A}$, as desired.

Now, consider any remaining type 3 chain. Lemma 12 and Corollary 2 ensure that it has a type 1 or type 0 chain on at least one side (right or left). By Lemma 13 and Corollary 3 and by the previous grouping of the chains of type 4, we know that this chain was not used by any group consisting of chains of type 4. Thus, every type 3 chain may be grouped with its adjacent type 1 or 0 chain. All remaining chains in the decomposition have at most 2 sets from $\mathcal{A}$ and so we may group them all together.

□

We now derive the LYM-type inequality, Theorem 19, from Lemma 11.

Proof. We will double count pairs (A, σ) where $A \in \mathcal{A}$ and σ is a cyclic permutation of $[n]$. Let $f(A, \sigma)$ be the indicator function for $A \in \mathcal{A}$ and A being an interval along σ. For each $A \in \mathcal{A}$, there are $|A|\,!\,(n - |A|)!$ cyclic permutations containing A as an interval. It follows that

$$\sum_{A \in \mathcal{A}} \sum_{\sigma} f(A, \sigma) = \sum_{A \in \mathcal{A}} |A|\,!\,(n - |A|)!\,.$$

On the other hand, Lemma 11 implies

$$\sum_{\sigma} \sum_{A \in \mathcal{A}} f(A, \sigma) \leq \sum_{\sigma} 2n = 2n!\,.$$

Dividing through by $n!$ gives

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 2,$$

as desired. □

Finally, we deduce Theorem 18 from Theorem 19.

Proof. If $\mathcal{A}$ contains neither $[n]$ nor $\varnothing$, then the result follows easily from Theorem 18. If $\mathcal{A}$ contains $[n]$, but there is an $n-1$ element set A not contained in $\mathcal{A}$, then replacing $[n]$ with A in $\mathcal{A}$ introduces no new relations and so yields another family of the same size without a skew-butterfly. Thus, in this case, Theorem 18 again yields the result. If $\mathcal{A}$ contains $[n]$ and the entire $(n-1)^{st}$ level, let $\mathcal{A}' = \{A \in \mathcal{A} : |A| \leq n-2\}$. Then, $\mathcal{A}'$ is an antichain, for otherwise we would have a skew-butterfly. Thus, $|\mathcal{A}'| \leq \binom{n}{\lfloor n/2 \rfloor}$ by Sperner's Theorem and so $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor} + n + 1$. For $n \geq 5$ this implies $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$. An analogous argument works for the case when $\varnothing \in \mathcal{A}$. If $n = 4$ we give another argument (we are still assuming $\mathcal{A}$ contains all $n-1$ element sets). If $\mathcal{A}'$ is a full level, then $\mathcal{A}$ contains a skew-butterfly. If $\mathcal{A}'$ is not a full level, then the equality case of Sperner's theorem implies $|\mathcal{A}'| \leq \binom{n}{\lfloor n/2 \rfloor} - 1$, and so $|\mathcal{A}| \leq n + \binom{n}{\lfloor n/2 \rfloor}$ which yields the required bound when $n = 4$. The case $n = 3$ is easily checked by hand. $\square$

We end this subsection by mentioning the relation between this approach and the double chain method. It is not hard to see that a double chain has the exact same poset structure as two consecutive chains in the chain decomposition described above. Namely, the degree 2 vertices from the Hasse diagram of consecutive chains correspond to the sets from the secondary chain of a double chain. It follows that any forbidden subposet result that can be determined exactly with the double chain method can also be determined exactly using a decomposition of a cyclic permutation, and, thus, chain decompositions of the cycle may be viewed as a generalization of the double chain method.

3.3 Forbidding Y_k and Y_k'

We will use the same decomposition of the cycle as in Subsection 3.2. The new bound we must prove is

Lemma 14. *If $\mathcal{A}$ is a collection of intervals along a cyclic permutation σ of $[n]$ which does not contain Y_k or Y_k' as a subposet, then*

$$|\mathcal{A}| \leq kn.$$

As before, we will consider groups of consecutive chains. Each chain, $\mathcal{C}$, with $k+1$ sets in $\mathcal{C} \cap \mathcal{A}$ is characterized by whether the second largest element in $\mathcal{A} \cap \mathcal{C}$ faces left or faces right (has even or odd cardinality, respectively). We say that a chain with $k+1$ elements of $\mathcal{A}$ is of type $(k+1)^R$ if the second largest element faces right and $k+1^L$ if it faces left.

Lemma 15. *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be consecutive chains in the decomposition. If $\mathcal{C}_i$ is of type $(k+1)^R$, then $|\mathcal{A} \cap \mathcal{C}_{i+1}| \leq k-1$, and $|\mathcal{A} \cap \mathcal{C}_{i+1}| = k-1$ implies that the largest element of $\mathcal{A} \cap \mathcal{C}_{i+1}$ is the same size as the second largest element of $\mathcal{A} \cap \mathcal{C}_i$.*

Proof. Let A be the second smallest set in $\mathcal{A} \cap \mathcal{C}_i$ and B be the second largest. Let Y be the set of size $|B|$ in $\mathcal{C}_{i+1}$, and if A is degree 2 (left), then let X be the set of size $|A|-1$ in $\mathcal{C}_{i+1}$. If A is degree

4, then let X be the set of size $|A|$ in $\mathcal{C}_{i+1}$. In either case, let $\mathcal{R}$ be the collection of those sets in $\mathcal{C}_{i+1}$ (not necessarily in $\mathcal{A}$) having sizes strictly between $|X|$ and $|Y|$ (see Figure 3.5). Every set in $\mathcal{C}_{i+1} \cap \mathcal{A}$ must lie in $\mathcal{R} \cup \{X\} \cup \{Y\}$ for otherwise we would have a Y_k or Y'_k or a $(k+2)$-chain. Now, $|\mathcal{A} \cap \mathcal{R}| \leq k-2$ for otherwise we would have a $k+2$ chain (actually, $|\mathcal{A} \cap \mathcal{R}| \leq k-3$ in the case A is degree 4). If we take $k-1$ sets from $\mathcal{R} \cup \{X\}$, then we have a Y'_k and so we can take at most $k-2$ sets total from $\mathcal{R} \cup \{X\}$. It follows that $|\mathcal{A} \cap \mathcal{C}_{i+1}| \leq k-1$ with equality only if $Y \in \mathcal{A}$. □

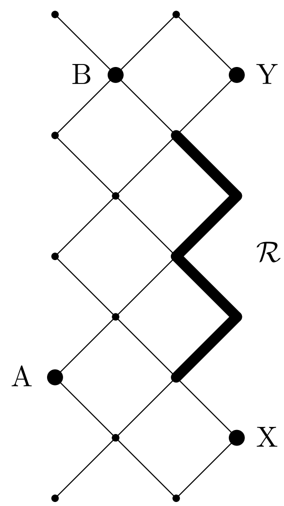

Figure 3.5: The sets A, B, X and Y are shown, and the collection $\mathcal{R}$ is marked in the case A is degree 2.

By a symmetric argument we have

Corollary 4. *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be consecutive chains in the decomposition. If $\mathcal{C}_{i+1}$ is of type $(k+1)^L$, then $|\mathcal{A} \cap \mathcal{C}_i| \leq k-1$, and $|\mathcal{A} \cap \mathcal{C}_i| = k-1$ implies that the largest element of $\mathcal{A} \cap \mathcal{C}_i$ is the same size as the second largest element of $\mathcal{A} \cap \mathcal{C}_{i+1}$.*

Lemma 16. *There are no 3 consecutive chains $\mathcal{C}_i, \mathcal{C}_{i+1}, \mathcal{C}_{i+2}$ such that $\mathcal{C}_i$ is type $(k+1)^R$, $\mathcal{C}_{i+1}$ is type $k-1$ and $\mathcal{C}_{i+1}$ is type $(k+1)^L$.*

Proof. Since $\mathcal{C}_i$ is type $(k+1)^R$ and $\mathcal{C}_{i+2}$ is type $(k+1)^L$, the respective second largest elements of $\mathcal{A} \cap \mathcal{C}_i$ and $\mathcal{A} \cap \mathcal{C}_{i+2}$ must be of different sizes (we argued before that we may assume they are degree 2 vertices, and the degree two vertices in $\mathcal{C}_i$ and $\mathcal{C}_{i+2}$ have sizes of different parity). It follows from Lemma 15 and Corollary 4 that we can have at most $k-2$ sets in $\mathcal{A} \cap \mathcal{C}_{i+1}$. □

We now have what we need to prove Lemma 14.

Proof of Lemma 14. Every group of 3 consecutive chains of type $(k+1)^R$, $\leq k-2$ and $(k+1)^L$, respectively, may be grouped together yielding a total of at most $3k$ sets on 3 chains. All remaining chains of type $(k+1)^R$ may be paired with a chain of at most $k-1$ sets from $\mathcal{A}$ following it, and all remaining chains of type $(k+1)^L$ may be paired with a chain of at most $k-1$ sets preceding it. It follows that $\mathcal{A}$ consists of at most kn intervals along the cyclic permutation σ. □

Theorem 21 follows directly from Lemma 14 as before. It remains to use Theorem 21 to deduce Theorem 20.

Proof of Theorem 20. Let $\mathcal{A} \subset 2^{[n]}$ be a Y_k and Y_k'-free family. If neither of $\varnothing$ and $[n]$ are in $\mathcal{A}$, then the result is immediate from Theorem 21. If $\varnothing$ and $[n]$ are in $\mathcal{A}$, then $\mathcal{A} \setminus \{\varnothing, [n]\}$ is k-chain free and so has size at most $\Sigma(n, k-1)$ by Erdős's theorem. Since $2 + \Sigma(n, k-1) \leq \Sigma(n, k)$ for $n \geq k+1$ and $k \geq 2$, we are done. Finally, suppose that $\varnothing \in \mathcal{A}$ and $[n] \notin \mathcal{A}$. If there is a singleton set $\{x\} \notin \mathcal{A}$, then we may replace $\varnothing$ with $\{x\}$ and we are back in the first case. Hence, we may assume that $\mathcal{A}$ contains every singleton set $(\binom{[n]}{1} \subset \mathcal{A})$. Let $\mathcal{A}' = \mathcal{A} \setminus \{\{\varnothing\} \cup \binom{[n]}{1}\}$. Now, $\mathcal{A}'$ is k-chain free, so again by Erdős's theorem, $|\mathcal{A}'| \leq \Sigma(n, k-1)$. It follows that $|\mathcal{A}| \leq 1 + n + \Sigma(n, k-1)$. If $\mathcal{A}'$ contains $k-1$ full levels, then we have a copy of Y_k', so we may assume we do not. However, then we may apply the equality case of Erdős's theorem to obtain that $|\mathcal{A}| \leq n + \Sigma(n, k-1)$. Finally, since $n \geq k+1$ implies that the k^{th} largest level has size at least n, we have $|\mathcal{A}| \leq \Sigma(n, k)$, as desired. □

3.4 Forbidding induced Y and Y' and second cycle decomposition

As in the proof of Theorem 18, we will need to prove a lemma which bounds the largest intersection of a family without an induced Y or Y' with the set of intervals along a cyclic permutation.

Lemma 17. *If $\mathcal{A}$ is a collection of intervals along a cyclic permutation σ of $[n]$ which does not contain Y or Y' as an induced subposet, then*

$$|\mathcal{A}| \leq 2n.$$

Proof. We will consider a different way of partitioning the chains along σ from the one in the proofs of the previous theorems. Let σ be the ordering $x_1, x_2, \ldots, x_n, x_1$. Group the intervals along σ into chains $\mathcal{C}_i = \{\{x_i\}, \{x_i, x_{i+1}\}, \{x_i, x_{i+1}, x_{i+2}\}, \ldots, \{x_i, x_{i+1}, \ldots, x_{i+n-1}\}\}$ where $1 \leq i \leq n$. Observe that $\{\mathcal{C}_i\}_{i=1}^n$ is a partition of the intervals along σ.

We now consider a second way of partitioning the intervals by setting $\mathcal{C}'_i = \{\{x_i\}, \{x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i-2}\}, \ldots, \{x_i, x_{i-1}, \ldots, x_{i-n+1}\}\}$ for $1 \leq i \leq n$. Observe that $\{\mathcal{C}'_i\}_{i=1}^n$ is again a partition (see Figure 3.6).

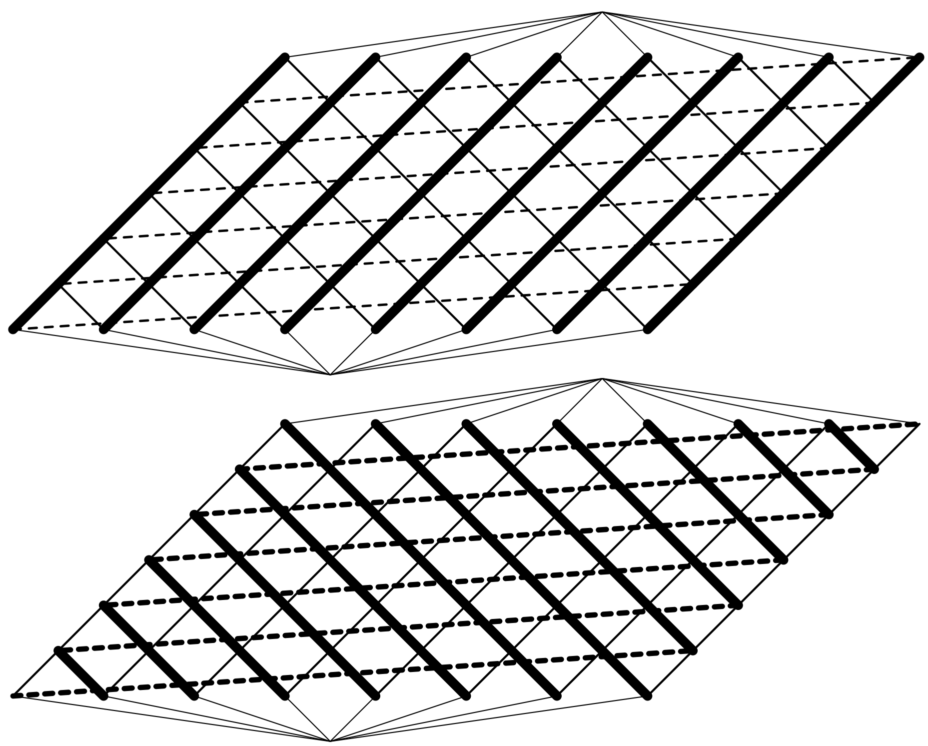

Figure 3.6: Orthogonal chain decompositions $\{\mathcal{C}_i\}_{i=1}^n$ (above) and $\{\mathcal{C}'_i\}_{i=1}^n$ (below) of the cycle are highlighted with bold lines. Dashed lines indicate how the chains wrap around.

Now, the two partitions we have defined have the property that if A and B are in $\mathcal{C}_i$ for some i, then at most one of A and B are in any $\mathcal{C}'_j$. Moreover, since each A is contained in exactly one chain in each partition, it follows that each A is contained in exactly 2 chains in the union of the two partitions. Thus, we have

$$\sum_{\mathcal{C} \in \{\mathcal{C}_i\}_{i=1}^n \cup \{C'_i\}_{i=1}^n} |\mathcal{A} \cap \mathcal{C}| = 2\,|\mathcal{A}|\,.$$

On the other hand, if a chain $\mathcal{C} \in \{\mathcal{C}_i\}_{i=1}^n$ intersects $\mathcal{A}$ in $k > 2$ sets $A_1, A_2, \ldots, A_k$ with $A_1 \subset A_2 \subset \ldots \subset A_k$, then there are $k-2$ chains in $\mathcal{C}' \in \{\mathcal{C}'_i\}_{i=1}^n$ such that $|\mathcal{A} \cap \mathcal{C}'| = 1$, namely those chains in $\{\mathcal{C}'_i\}_{i=1}^n$ containing $A_2, A_3, \ldots, A_{k-2}$ or A_{k-1}, as an intersection of greater than one would yield an induced Y or Y'. Similarly, if a chain $\mathcal{C}' \in \{\mathcal{C}'_i\}_{i=1}^n$ intersects $\mathcal{A}$ in $k > 2$ sets, then there are $k-2$ chains from $\{\mathcal{C}_i\}_{i=1}^n$ which intersect $\mathcal{A}$ in exactly one set. Here, we are using an additional property of the decomposition that if $A \in \mathcal{C} \cap \mathcal{C}'$, then no set larger than A in $\mathcal{C}$ is comparable to a set larger than A in $\mathcal{C}'$, and, similarly, no set smaller than A in $\mathcal{C}$ is comparable to a set smaller than A in $\mathcal{C}'$. We have shown that there is a total of $2k-2$ incidences of $\mathcal{A}$ with these $k-1$ chains. It follows, then, by grouping each chain which intersects $\mathcal{A}$ in at least 3 positions with a collection of chains that can only intersect $\mathcal{A}$ in one position, that the number of pairs $(A, \mathcal{C})$ where $A \in \mathcal{A}, \mathcal{C} \in \{\mathcal{C}_i\}_{i=1}^n \cup \{\mathcal{C}'_i\}_{i=1}^n$ and $A \in \mathcal{C}$ is at most twice the number of chains. Thus,

$$\sum_{\mathcal{C} \in \{\mathcal{C}_i\}_{i=1}^n \cup \{C'_i\}_{i=1}^n} |\mathcal{A} \cap \mathcal{C}| \leq 2 \left| \{\mathcal{C}_i\}_{i=1}^n \cup \{C'_i\}_{i=1}^n \right| = 4n.$$

Dividing through by 2 yields the desired inequality. $\square$

Lemma 17 implies the LYM-type inequality, Theorem 23, exactly as in the previous proofs. It remains to derive the bound on $\mathrm{La}^{\#}(n, Y, Y')$ using Theorem 23.

Proof of Theorem 22. If $\mathcal{A}$ contains neither $\varnothing$ nor $[n]$, then we are done by Theorem 23. If $\varnothing$ and $[n]$ are in $\mathcal{A}$, then $\mathcal{A} \setminus \{\varnothing, [n]\}$ is induced V and Λ free. It follows from Katona and Tarjan [51] that

$$\mathrm{La}^{\#}(n, Y, Y') \leq 2 + \mathrm{La}^{\#}(n, V, \Lambda) = 2 + 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} \leq \Sigma(n, 2).$$

Now, assume without loss of generality that $\varnothing \notin \mathcal{A}$ but $[n] \in \mathcal{A}$, and let $\mathcal{A}' = \mathcal{A} \setminus \{[n]\}$. Since $\mathcal{A}'$ is induced Y and Y'-free, it satisfies the hypothesis of Theorem 23. Assume, by contradiction, that $|\mathcal{A}'| = \Sigma(n, 2)$. It follows that equality holds in Theorem 23. If n is odd, then we have that $\mathcal{A}' = \binom{[n]}{\lfloor n/2 \rfloor} \cup \binom{[n]}{\lceil n/2 \rceil}$ which implies $\mathcal{A}$ induces a Y', contradiction. If n is even, then $\binom{[n]}{n/2} \subset \mathcal{A}'$ and $\binom{n}{n/2+1}$ sets from $\binom{[n]}{n/2-1} \cup \binom{[n]}{n/2+1}$ are in $\mathcal{A}$. Since $\mathcal{A}$ contains no induced Y', it follows that $\mathcal{A}' \cap \binom{[n]}{n/2+1} = \varnothing$. Thus, we must have $\mathcal{A}' = \binom{[n]}{n/2-1} \cup \binom{[n]}{n/2}$, but then $\mathcal{A}$ still contains an induced Y', contradiction.

$\square$

Chapter 4

The maximum size of intersecting P-free families

In the course of determining the profile polytope for complement-free k-Sperner families, Gerbner [32] proved a generalization of Milner's theorem (the 1-intersecting case) to the k-Sperner setting.

Theorem 24 (Gerbner [32]). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting k-Sperner family, then*

$$|\mathcal{F}| \leq \begin{cases} \sum\limits_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i}, & \text{if } n \text{ is odd} \\ \binom{n-1}{\frac{n}{2}-1} + \sum\limits_{i=\frac{n}{2}+1}^{\frac{n}{2}+k-1} \binom{n}{i} + \binom{n-1}{\frac{n}{2}+k}, & \text{if } n \text{ is even.} \end{cases} \tag{4.1}$$

For simplicity, we denote the right hand side of (4.1) by $\sum_I(n,k)$. For any given P, we define

$$\mathrm{La}_I(n,P) = \max_{\mathcal{F} \subseteq 2^{[n]}} \{|\mathcal{F}| : \mathcal{F} \text{ does not contain } P \text{ as a subposet and } \mathcal{F} \text{ is intersecting}\}.$$

In this language, Theorem 24 states that $\mathrm{La}_I(n,P_{k+1}) = \sum_I(n,k)$, where P_{k+1} is the path poset of length $k+1$. Before we state our main results we need to introduce some notation. For all n and $k \leq n/2$, define

$$\mathcal{H}_{0,n,k} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 1} \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 2} \cup \ldots \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + k}$$

and in the case when n is even, for any $x \in [n]$, define

$$\mathcal{H}_{x,n,k} = \{F : F \in \binom{[n]}{\frac{n}{2}} : x \in F\} \cup \binom{[n]}{\frac{n}{2}+1} \cup \ldots \cup \binom{[n]}{\frac{n}{2}+k-1} \cup \{F : F \in \binom{[n]}{\frac{n}{2}+k} : x \notin F\}.$$

We determine the exact value of $\mathrm{La}_I(n, B)$, the maximum size of an intersecting butterfly-free family, for $n \geq 18$. In particular, we show that $\mathrm{La}_I(n, B) = \Sigma_I(n, 2)$. The cases of equality are also obtained.

Theorem 25 (Gerbner, Methuku, T)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting B-free family of subsets of $[n]$ where $n \geq 18$. Then,*

$$|\mathcal{F}| \leq \Sigma_I(n, 2).$$

Equality holds if and only if:

- *For n odd, $\mathcal{F} = \mathcal{H}_{0,n,2}$;*
- *For n even, $\mathcal{F} = \mathcal{H}_{x,n,2}$ for some $x \in [n]$.*

The proof of this theorem can be seen as a generalization of the partition method of Griggs, Li and Lu [39, 37] to a weighted setting involving cyclic permutations. We also show that a variant of the LYM-type inequalities, Theorems 9 and 10, hold in this case.

Theorem 26 (Gerbner, Methuku, T)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting B-free family of sets F such that $2 \leq |F| \leq n-2$, then*

$$\sum_{\substack{F \in \mathcal{F} \\ |F| \leq \frac{n}{2}}} \frac{1}{\binom{n}{|F|-1}} + \sum_{\substack{F \in \mathcal{F} \\ |F| > \frac{n}{2}}} \frac{1}{\binom{n}{|F|}} \leq 2.$$

Theorem 27 (Gerbner, Methuku, T)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting B-free family, and assume that for all $F \in \mathcal{F}$ we have $2 \leq |F| \leq n/2$, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-1}{|F|-1}} \leq 2.$$

Next we obtain an upper bound on $\mathrm{La}_I(n, P)$ for an arbitrary poset P in the case when n is odd. Let $h(P)$ be the height of the poset P, that is, the size of the longest chain in P.

Theorem 28 (Gerbner, Methuku, T)**.** *Assume n is odd and $\frac{|P|+h(P)}{2}$ is an integer. Let $\mathcal{F}$ be an intersecting P-free family of subsets of $[n]$ $(n \geq 4)$. Then,*

$$|\mathcal{F}| \leq \sum_{i=1}^{\frac{|P|+h(P)}{2}-1} \binom{n}{\lfloor \frac{n}{2} \rfloor + i}.$$

Note 2. *Let $e(P)$ denote the maximum number of consecutive levels in $2^{[n]}$ which do not contain a copy of P as a subposet for any n. Recall that Burcsi and Nagy (in Theorem 12) determined the exact value of $\mathrm{La}(n, P)$ for infinitely many posets P for which $e(P) = \frac{|P|+h(P)}{2} - 1$. For all these posets we have equality in Theorem 28. In the cases where n is even or $\frac{|P|+h(P)}{2}$ is not an integer, a similar bound can be obtained, but it is not sharp in general.*

Finally, we give a new proof of Theorem 24 which avoids the usage of profile polytopes. We also classify the cases of equality.

Theorem 29 (Gerbner, Methuku, T (equality cases))**.** *Let $\mathcal{F}$ be an intersecting k-Sperner family of subsets of $[n]$. Then,*

$$|\mathcal{F}| \leq \Sigma_I(n, k).$$

If $k \leq n/2$, then equality holds in the following cases:

- *For n odd, $\mathcal{F} = \mathcal{H}_{0,n,k}$;*
- *For n even and $k = 1$, $\mathcal{F} = \mathcal{H}_{0,n,k}$ or $\mathcal{H}_{x,n,k}$ for some $x \in [n]$;*
- *For n even and $k > 1$, $\mathcal{F} = \mathcal{H}_{x,n,k}$ for some $x \in [n]$.*

Note 3. *If $k > n/2$ there can be many extremal families. For example, if n is even and $k = n/2+1$, we may take any intersecting family on level $n/2$ (of which there are many) in addition to all complete levels from $n/2+1$ to n. Note, however, that the inequality portion of our result holds for all n and k.*

The chapter is organized as follows. In Subsection 4.1 we prove Theorem 29 about intersecting k-Sperner families. In Subsection 4.2 we prove Theorem 25 determining the exact value of $La_I(n, B)$. In Subsection 4.3 we prove Theorem 26 and Theorem 27. Finally, in the last subsection we prove Theorem 28 about general posets P.

4.1 Intersecting k-Sperner families

The aim of this subsection is to prove Theorem 29. Recall that a *cyclic permutation* of $[n]$ (in the sense of Katona [45]) is an arrangement of the numbers 1 through n along a circle. Sets of consecutive elements along the circle are called *intervals*. The proof will proceed by double counting pairs (F,σ) with a weight function where $F \in \mathcal{F}$ and σ is a cyclic permutation. For any collection $\mathcal{H}$ of sets, let $\mathcal{H}^\sigma = \{F : F \in \mathcal{H} \text{ and } F \text{ is an interval along } \sigma\}$. The collection of all intervals along σ of size r is denoted $\mathcal{L}_r^\sigma$. Let $\mathcal{F}$ be an intersecting k-Sperner family. We will also assume $[n] \notin \mathcal{F}$, for otherwise we may apply a simple inductive argument. In the double counting we will use the following weight function:

$$w(F,\sigma) = \begin{cases} \binom{n}{|F|}, & \text{if } F \in \mathcal{F} \text{ and } F \text{ is an interval along } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

Observe that, on the one hand, we have

$$\sum_{F\in\mathcal{F}} \sum_{\sigma} w(F,\sigma) = \sum_{F\in\mathcal{F}} |F|\,!\,(n-|F|)! \binom{n}{|F|} = n!\,|\mathcal{F}|\,.$$

On the other hand,

$$\sum_{\sigma} \sum_{F\in\mathcal{F}} w(F,\sigma) = \sum_{\sigma} \sum_{F\in\mathcal{F}^\sigma} \binom{n}{|F|} \le (n-1)! \max_{\sigma} \sum_{F\in\mathcal{F}^\sigma} \binom{n}{|F|}.$$

We will show that $\sum_{F\in\mathcal{F}^\sigma} \binom{n}{|F|} \le n\Sigma_I(n,k)$ for all σ and determine the equality cases. Before proving Theorem 29 we must establish some preliminary facts about cyclic permutations.

For notational simplicity, we will often work with the simplest case of a cyclic permutation where the numbers $1,2,\ldots,n$ occur in that order. We call this cyclic permutation the *canonical* cyclic permutation. Along this permutation we let A_i^j denote the interval $\{i, i+1, \ldots, i+j-1\}$ (addition involving the base set is always taken modulo n). It is clear that when we are working with one fixed cyclic permutation we may assume it is canonical because renaming the elements will not change the intersection or containment structure of its intervals.

Lemma 18. *Let $\mathcal{G}$ be an antichain of intervals along a cyclic permutation σ, then $|\mathcal{G}| \le n$, and equality holds if and only if $\mathcal{G}$ consists of every interval of some size.*

Proof. Assume that σ is canonical. We partition all intervals along σ into chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$ where $\mathcal{C}_i = \{\{i\}, \{i, i+1\}, \ldots, \{i, i+1, \ldots, i+n-1\}\}$. Since at most one interval from each chain may be in our collection, we have that either we take fewer than n intervals or every chain contains exactly one interval from $\mathcal{G}$. Suppose we are in the latter case and that some two intervals in $\mathcal{G}$ had different sizes. Then, there must exist chains $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ where the interval we take in $\mathcal{C}_i$ is larger than the one we take in $\mathcal{C}_{i+1}$. That is, we have $A_i^{j_1}, A_{i+1}^{j_2} \in \mathcal{G}$ with $j_1 > j_2$. However, this implies that we have $A_{i+1}^{j_2} \subseteq A_i^{j_1}$, a contradiction. $\square$

Let σ be canonical, and $\mathcal{G}$ be a collection of intervals along σ. If $\mathcal{G}$ contains only intervals of size j of the form $A_i^j, A_{i+1}^j, \ldots, A_{i+s}^j$, then we say that $\mathcal{G}$ is *contiguous*. If $\mathcal{G}$ is a collection consisting of intervals $A_i^j, A_{i+1}^j, \ldots, A_{i+s}^j, A_{i+s+1}^{j+1}, A_{i+s+2}^{j+1}, \ldots, A_{i-2}^{j+1}$, then we say $\mathcal{G}$ is *pair-contiguous*. Equivalently, $\mathcal{G}$ is pair-contiguous if it is an antichain, has size $n-1$, and is the union of two contiguous collections of intervals spanning two consecutive levels. We extend these definitions to arbitrary cyclic permutations in the obvious way.

Lemma 19. *If $\mathcal{G}$ is an antichain of intervals along a cyclic permutation σ such that $|\mathcal{G}| = n-1$ and $\mathcal{G}$ contains intervals of at least two sizes, then $\mathcal{G}$ is pair-contiguous.*

Proof. Assume that σ is canonical. Let $\mathcal{G}_{min}$ be the collection of those intervals in $\mathcal{G}$ of minimum size, say j. Since $\mathcal{G}_{min}$ is not a full level there must be an i such that $A_i^j \in \mathcal{G}_{min}$ but $A_{i-1}^j \notin \mathcal{G}_{min}$. Then, we know that $\mathcal{C}_{i-1}$ has no interval from $\mathcal{G}$, and if $|\mathcal{G}| = n-1$ it must be that each chain $\mathcal{C}_i, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{i-2}$ contains an interval from $\mathcal{G}$. Observe that if $\mathcal{G}$ contains an interval of size j_1 in $\mathcal{C}_{i_1}$ and an interval of size j_2 in $\mathcal{C}_{i_1+1}$, then $j_1 \leq j_2$, for otherwise we would not have an antichain. Finally, the interval from $\mathcal{G}$ in $\mathcal{C}_{i-2}$ must have size $j+1$, for if it were any larger it would contain A_i^j. It follows that $\mathcal{G}$ is a pair-contiguous family contained in levels j and $j+1$. $\square$

Lemma 20. *Let $\mathcal{G}$ be an intersecting k-Sperner collection of intervals along a cyclic permutation σ, then*

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq n\Sigma_I(n, k). \tag{4.2}$$

Assume $k \leq n/2$, then equality holds in (4.2) if and only if:

- *n is odd and $\mathcal{G} = \mathcal{H}_{0,n,k}^{\sigma}$;*
- *n is even, $k = 1$ and $\mathcal{G} = \mathcal{H}_{0,n,1}^{\sigma}$ or $\mathcal{G} = \mathcal{H}_{x,n,1}^{\sigma}$ for some $x \in [n]$;*

- *n is even, $k > 1$ and $\mathcal{G} = \mathcal{H}^{\sigma}_{x,n,k}$ for some $x \in [n]$.*

Proof. First, fix k and suppose that n is odd. Following an argument of Mirsky [68], $\mathcal{G}$ can be decomposed into k antichains in the following way. For $1 \leq i \leq k$ set

$$\mathcal{G}_i = \{G : G \in \mathcal{G} \text{ and the longest chain in } \mathcal{G} \text{ with maximal element } G \text{ has length } i\}.$$

Now, we have for each i that $|\mathcal{G}_i| \leq n$ by the antichain property, and so $|G| \leq kn$.

For any interval G along σ, it is easy to see that $[n] \setminus G$ is also an interval along σ and has size $n - |G|$. Since our family is intersecting, by pairing off each G with $[n] \setminus G$, we see that $\mathcal{G}$ contains at most n intervals of size $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ (yielding the largest possible weight) and at most n intervals of size $\lfloor n/2 \rfloor - 1$ or $\lfloor n/2 \rfloor + 2$ (yielding the second largest possible weight) and so on. Thus, the bound

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq n\left(\binom{n}{\lfloor \frac{n}{2} \rfloor + 1} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 2} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor + k}\right) = n\Sigma_I(n,k)$$

is immediate. Assume now that $\mathcal{G}$ attains this weight and $k \leq n/2$, then $\mathcal{G}$ must contain n sets from each of $\mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor} \cup \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor+1}$, $\mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor-1} \cup \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor+2}$, $\ldots$, $\mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor-k+1} \cup \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor+k}$. In particular, we must have $|\mathcal{G}| = kn$.

Observe that each $\mathcal{G}_i$ is an antichain and, since $|\mathcal{G}| = kn$, we have $|\mathcal{G}_i| = n$ for all i. Then, Lemma 18 implies that each $\mathcal{G}_i$ is equal to a level of intervals along σ. If $\mathcal{G}_i$ is a level, it must consist of intervals of size at least $\lfloor n/2 \rfloor + 1$. Thus, we have

$$\mathcal{G}_i = \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor + i}$$

for each i and so

$$\mathcal{G} = \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor+1} \cup \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor+2} \cup \ldots \cup \mathcal{L}^{\sigma}_{\lfloor \frac{n}{2} \rfloor+k} = \mathcal{H}^{\sigma}_{0,n,k}.$$

Next, we consider the case when n is even and $k = 1$. By Lemma 18, if $|\mathcal{G}| = n$, then $\mathcal{G}$ is a level $\mathcal{L}^{\sigma}_i$ for some i. By the intersection property, we have $i \geq n/2 + 1$ and so the weight of the family is bounded by $n\binom{n}{n/2+1}$ with equality only if $\mathcal{G} = \mathcal{L}^{\sigma}_{\frac{n}{2}+1}$. If $|\mathcal{G}| \leq n - 1$ then, since we can take at most $n/2$ intervals of size $n/2$, the weight is bounded by $\frac{n}{2}\binom{n}{\frac{n}{2}} + (\frac{n}{2} - 1)\binom{n}{\frac{n}{2}+1}$. This bound can only be attained if $|\mathcal{G}| = n - 1$, and it follows by Lemma 19 that G is pair-contiguous which, in the case $k = 1$, implies $\mathcal{G} = \mathcal{H}^\sigma_{x,n,1}$ for some $x \in [n]$. Since $\frac{n}{2}\binom{n}{\frac{n}{2}} + (\frac{n}{2} - 1)\binom{n}{\frac{n}{2}+1} = n\binom{n}{\frac{n}{2}+1}$, both the $|\mathcal{G}| = n - 1$ case and the $|\mathcal{G}| = n$ case yield optimal configurations.

Finally, we consider the case when n is even and $k > 1$. Suppose first that none of $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are levels. Then, by Lemma 18 we have $|\mathcal{G}_i| \leq n-1$ for all i. We have $|\mathcal{G}| \leq kn - k$, and we will see that if $\mathcal{G}$ has maximal weight, then in fact $|\mathcal{G}| \geq kn - k$. Indeed, by pairing off intervals with their complements, we can have at most $n/2$ intervals of size $n/2$, n intervals of size $n/2 - 1$ or $n/2 + 1$ and so on. Thus, the total weight we can achieve with $kn - k$ intevals is bounded by

$$\frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}+k-1} + \left(\frac{n}{2} - k\right)\binom{n}{\frac{n}{2}+k} = n\Sigma_I(n,k),$$

and if we have fewer than $kn - k$ intervals the weight will be strictly less than this. It follows that we may assume $|\mathcal{G}| = kn - k$ and $|\mathcal{G}_i| = n - 1$ for all $1 \leq i \leq k$. By Lemma 19 each $\mathcal{G}_i$ is pair-contiguous on two levels j and $j + 1$. If $j < n/2$, then the corresponding $\mathcal{G}_i$ would have size at most $n/2$ (from Katona's proof of Theorem 5 [29]). Thus, we may assume that $j \geq n/2$. However, this combined with the fact that $\mathcal{G}$ has maximal weight already determines the structure of $\mathcal{G}$. Namely, $\mathcal{G}_1$ is pair-contiguous spanning levels $n/2$ and $n/2 + 1$ with $n/2$ sets of size $n/2$ forming a star about some element x, $\mathcal{G}_2$ is pair-contiguous spanning levels $n/2 + 1$ and $n/2 + 2$ containing all remaining $n/2 + 1$ elements of $\mathcal{L}^\sigma_{\frac{n}{2}+1}$ and a contiguous part of $\mathcal{L}^\sigma_{\frac{n}{2}+2}$ ($\mathcal{G}_2$ contains all the $n/2+1$ element intervals that contain x) and so on. It follows that $\mathcal{G} = \mathcal{H}^\sigma_{x,n,k}$ for some $x \in [n]$.

Now, we will show that if $\mathcal{G}$ has maximal weight, then it cannot be that any of the $\mathcal{G}_i$ are levels. This will complete the proof since we have already classified the extremal families in the case that there are no levels. Suppose, by way of contradiction, that s is the smallest number such that $\mathcal{G}_s$ is a level, say $\mathcal{L}^\sigma_t$ $(t > n/2)$. The weight of $\mathcal{G}_s \cup \mathcal{G}_{s+1} \cup \ldots \cup \mathcal{G}_k$ is clearly bounded by

$$n\binom{n}{t} + n\binom{n}{t+1} + \cdots + n\binom{n}{t+k-s}.$$

If $t > n/2 + s - 1$, then, by the previous case (no full levels), the weight of $\mathcal{G}_1 \cup \ldots \mathcal{G}_{s-1}$ is maximized by taking

$$\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1} = \mathcal{H}^\sigma_{x,n,s-1},$$

for some $x \in [n]$. The weight of $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1}$ is

$$w(\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1}) = \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}+s-2} + \left(\frac{n}{2} - (s-1)\right)\binom{n}{\frac{n}{2}+s-1},$$

and it follows that the total weight of $\mathcal{G}$ is at most

$$\begin{aligned}&\frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}+s-2}\\ &\qquad + \left(\frac{n}{2} - (s-1)\right)\binom{n}{\frac{n}{2}+s-1} + n\binom{n}{\frac{n}{2}+s} + n\binom{n}{\frac{n}{2}+s+1} + \cdots + n\binom{n}{\frac{n}{2}+k}. \qquad (4.3)\end{aligned}$$

Subtracting $w(\mathcal{H}^{\sigma}_{x,n,k}) - w(\mathcal{G})$ we obtain

$$\begin{aligned}w(\mathcal{H}^{\sigma}_{x,n,k}) - w(\mathcal{G}) &\geq \left(\frac{n}{2} + s - 1\right)\binom{n}{\frac{n}{2}+s-1} - \left(\frac{n}{2} + k\right)\binom{n}{\frac{n}{2}+k}\\ &= n\left(\binom{n-1}{\frac{n}{2}+s-1} - \binom{n-1}{\frac{n}{2}+k-1}\right) > 0, \qquad (4.4)\end{aligned}$$

which implies $\mathcal{G}$ does not have maximum weight.

Next, consider the case when $t \leq n/2 + s - 1$. By pairing off intervals with their complement along σ, it follows that

$$w(\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1}) \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{t-1}.$$

Thus, the whole weight is

$$w(\mathcal{G}) \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{t-1} + n\binom{n}{t} + \cdots + n\binom{n}{t+k-s},$$

but $t - s \leq n/2 - 1$ so

$$w(\mathcal{G}) \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}-1+k} < w(\mathcal{H}^{\sigma}_{x,k,n}).$$

Thus, we may conclude that there is no full level. It follows that the only possible equality case is $\mathcal{G} = \mathcal{H}^{\sigma}_{x,k,n}$ for some $x \in [n]$.

$\square$

Proof of Theorem 29. By Lemma 4.2 we have that for every σ,

$$\sum_{F\in\mathcal{F}^\sigma} w(F,\sigma) \le n\Sigma_I(n,k). \tag{4.5}$$

By the double counting outlined at the beginning of the subsection, it is immediate that $|\mathcal{F}| \le \Sigma_I(n,k)$. Thus, it remains to determine the possible extremal families. If $\mathcal{F}$ is extremal, then for every σ we have equality in (4.5) and so we are in an equality case given by Lemma 4.2.

Assume first that n is odd, then for every σ we have that $\mathcal{F}^\sigma$ is equal to $\mathcal{H}^\sigma_{0,n,k}$. In this case, it is immediate that $\mathcal{F} = \mathcal{H}_{0,n,k}$.

Suppose now that n is even and $k = 1$. There are two cases: either there exists a σ for which $\mathcal{F}^\sigma = \mathcal{H}^\sigma_{0,n,1}$ or there does not. Assume that we have $\mathcal{F}^\sigma = \mathcal{H}^\sigma_{0,n,1}$, and form a new cyclic permutation σ' by transposing two adjacent elements of σ. Observe that $\mathcal{F}^{\sigma'}$ still contains $n-2$ of the same intervals on level $n/2+1$ (namely, those without exactly one of the transposed elements). Now, configurations of the form $\mathcal{H}^\sigma_{x,n,1}$, $x \in [n]$, have $n/2-1$ intervals of size $n/2+1$. Thus, we have that $\mathcal{F}^{\sigma'}$ must have the form $\mathcal{H}^{\sigma'}_{0,n,1}$. Since every permutation can be generated by transpositions of consecutive elements it follows that for all σ, $\mathcal{F}^\sigma = \mathcal{H}^\sigma_{0,n,1}$ and so $\mathcal{F} = \mathcal{H}_{0,n,1}$. Thus, we will assume that for all σ we have $\mathcal{F}^\sigma = \mathcal{H}^\sigma_{x,n,1}$, $x \in [n]$.

If n is even and $k > 1$ and $\mathcal{F}^\sigma = \mathcal{H}^\sigma_{0,n,k}$ for some σ, then in a completely analogous way to the above $k = 1$ case we can deduce that $\mathcal{F} = \mathcal{H}_{0,n,k}$. However, for $k > 1$ we have $|\mathcal{H}_{0,n,k}| < |\mathcal{H}_{x,n,k}|$. Indeed, simply observe

$$\begin{aligned}
|\mathcal{H}_{x,n,k}| - |\mathcal{H}_{0,n,k}| &= \binom{n-1}{\frac{n}{2}-1} + \binom{n-1}{\frac{n}{2}+k} - \binom{n}{\frac{n}{2}+k} \\
&= \binom{n-1}{\frac{n}{2}-1} - \binom{n-1}{\frac{n}{2}+k-1} \\
&> 0.
\end{aligned}$$

Thus, we may rule out the $\mathcal{H}_{0,n,k}$ case for $k > 1$ and conclude that $\mathcal{F}^\sigma \ne \mathcal{H}^\sigma_{0,n,k}$ for any σ.

So, finally, we may suppose that n is even and $k \ge 1$ and that for every σ, we have $\mathcal{F}^\sigma = \mathcal{H}^\sigma_{x,n,k}$ for some $x \in [n]$. We want to show that $\mathcal{F} = \mathcal{H}_{x,n,k}$ for some x. Each cyclic permutation contains $n/2$ intervals of size $n/2$ (from $\mathcal{H}^\sigma_{x,n,k}$) and n intervals of size $n/2+i$ for $1 \le i \le k-1$ and $\frac{n}{2} - k$

intervals of size $\frac{n}{2}+k$. By the transposition argument that we used above, we can easily show that all the sets of $\binom{[n]}{\frac{n}{2}+i}$ for $1 \le i \le k-1$ are in $\mathcal{F}$. It only remains to show that $\mathcal{F}$ contains all the sets of size $\frac{n}{2}$ that contain a fixed element and all the sets of size $\frac{n}{2}+k$ that don't contain that fixed element.

We supposed that for each σ, $\mathcal{F}^\sigma$ contains all of the $n/2$-element intervals containing some x and all the $(n/2+k)$-element intervals not containing that x. However, the x's corresponding to different σ's may be different. Our aim is to show that this is impossible. First, let us fix a cyclic permutation σ and notice that we have two sets A and B that are intervals along this σ and intersect in a single element, x. Suppose by contradiction that there exists an $n/2$-element set C (in $\mathcal{F}$) not containing x. Observe that $|[n] \setminus (A \cup B)| = 1$, and let us define $[n] \setminus (A \cup B) = \{y\}$. If C contains y, then we can find a cyclic permutation σ where A, B and C are intervals, a contradiction. However, if C does not contain y, we can find a cyclic permutation where A, B and $C \cup \{y\}$ are intervals. Along this σ, since we have two intervals (namely, A and B) that intersect just in x, all the $n/2$-element intervals must also contain x and all the $(n/2+k)$-element intervals do not contain x. In particular, there is an $(n/2+k)$-element interval, say K which contains $C \cup \{y\}$ (this is because the interval $C \cup \{y\}$ doesn't contain x). Now, since all the intervals along σ of sizes $n/2+i$, $2 \le i \le k-1$ are in $\mathcal{F}$, it is easy to find a $(k+1)$-chain in $\mathcal{F}$ consisting of C, $C \cup \{y\}$ and K, a contradiction. Thus, we can conclude that every $n/2$-element set C in $\mathcal{F}$ must contain x.

By a standard double counting of pairs (F, σ) where $F \in \mathcal{F}$ and F is an interval along σ, we can see that $\mathcal{F}$ contains exactly $\binom{n-1}{\frac{n}{2}-1}$ sets of size $n/2$, and by the previous paragraph all the $n/2$-sets in $\mathcal{F}$ must contain a fixed element. Therefore, $\mathcal{F}$ contains every $n/2$-element set containing a fixed element and nothing else. But this means $\mathcal{F}$ cannot contain any set of size $n/2+k$ containing x because otherwise we will have a $(k+1)$-chain in $\mathcal{F}$, a contradiction. But by the same double counting argument we can see that $\mathcal{F}$ contains $\binom{n-1}{n/2+k}$ sets of size $n/2+k$, and all these sets must not contain x. This shows that $\mathcal{F} = \mathcal{H}_{x,n,k}$, as desired, and we have established all the cases of equality for intersecting k-Sperner families.

$\square$

4.2 Intersecting B-free families

In this subsection we prove Theorem 25 by determining the exact value of $\mathrm{La}_I(n, B)$ and classifying the extremal families.

We may assume that $[n] \notin \mathcal{F}$. Indeed, if $[n] \in \mathcal{F}$, then $\mathcal{F} \setminus \{[n]\}$ contains no three sets A, B, C with $A, B \subset C$. In this case, a result of Katona and Tarján [51] shows that such a family may have size at most $(1 + 2/n)\binom{n}{\lfloor n/2 \rfloor}$. Thus, for $n \geq 7$ the family will be too small.

As in Subsection 4.1, we will consider pairs (F, σ) with the weight function

$$w(F, \sigma) = \begin{cases} \binom{n}{|F|}, & \text{if } F \in \mathcal{F} \text{ and } F \text{ is an interval along } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

As before, we have

$$\sum_{F \in \mathcal{F}} \sum_{\sigma} w(F, \sigma) = n! \, |\mathcal{F}| \, . \tag{4.6}$$

On the other hand,

$$\sum_{\sigma} \sum_{F \in \mathcal{F}} w(F, \sigma) = \sum_{\sigma} \sum_{F \in \mathcal{F}^{\sigma}} \binom{n}{|F|}. \tag{4.7}$$

Let $\mathcal{F}_m = \{F \in \mathcal{F} \mid \exists A, B \in \mathcal{F} \text{ such that } A \subset F \subset B\}$ (notice that A and B are unique since $\mathcal{F}$ is butterfly-free). We refer to $\mathcal{F}_m$ as the collection of *middle* sets in $\mathcal{F}$. Fix a cyclic permutation σ. We will distinguish four kinds of intervals in $\mathcal{F}^{\sigma}$ which we refer to as the *middle, isolated, top* and *bottom* intervals along σ.

$$\mathcal{M}_{\sigma} = \{F : F \in \mathcal{F}^{\sigma} \text{ and there exists } A, B \in \mathcal{A}^{\sigma} \text{ such that } A \subset F \subset B\};$$
$$\mathcal{I}_{\sigma} = \{F : F \in \mathcal{F}^{\sigma} \text{ and } F \text{ is comparable with no other interval in } \mathcal{F}^{\sigma}\ \};$$
$$\mathcal{T}_{\sigma} = \{F : F \in \mathcal{F}^{\sigma} \setminus \mathcal{I}_{\sigma} \text{ is inclusion maximal in } \mathcal{F}^{\sigma}\};$$
$$\mathcal{B}_{\sigma} = \{F : F \in \mathcal{F}^{\sigma} \setminus \mathcal{I}_{\sigma} \text{ is inclusion minimal in } \mathcal{F}^{\sigma}\}.$$

It is easy to see that these four sets of intervals form a partition of $\mathcal{F}^{\sigma}$. Importantly, note that the four collections are defined by their properties as intervals along σ, not in $\mathcal{F}$ itself. So we may have, for example, a set $F \in \mathcal{F}_m$ which is an interval along σ, but does not belong to $\mathcal{M}_{\sigma}$.

For any $F \in \mathcal{F}$, let α_F be the number of cyclic permutations containing F as a middle interval and β_F be the number of cyclic permutations containing F as an isolated interval. Our proof considers the tradeoffs associated with these two possibilities. We will need to know the relative frequency with which they occur. To this end, define

$$c = \max_{F \in \mathcal{F}_m} \frac{\alpha_F}{\beta_F}.$$

For a fixed cyclic permutation σ, let $m_\sigma, i_\sigma, t_\sigma$ and b_σ denote the weight of the collections $\mathcal{M}_\sigma, \mathcal{I}_\sigma, \mathcal{T}_\sigma$ and $\mathcal{B}_\sigma$ respectively. Define

$$R = n\Sigma_I(n,2) = \begin{cases} n\binom{n}{\lfloor \frac{n}{2} \rfloor + 1} + n\binom{n}{\lfloor \frac{n}{2} \rfloor + 2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}\binom{n}{2} + n\binom{n}{\frac{n}{2}+1} + \left(\frac{n}{2} - 2\right)\binom{n}{\frac{n}{2}+2}, & \text{if } n \text{ is even.} \end{cases}$$

Thus, our aim is to show $|\mathcal{F}| \leq R/n$.

Lemma 21. *If for each cyclic permutation σ we have $t_\sigma + b_\sigma + (1+c)i_\sigma \leq R$, then $|\mathcal{F}| \leq R/n$.*

Proof. By (4.6) and (4.7), it suffices to show that

$$n!\,|\mathcal{F}| = \sum_\sigma \sum_{F \in \mathcal{F}^\sigma} \binom{n}{|F|} \leq (n-1)!\,R.$$

For a given σ we have

$$\sum_{F \in \mathcal{F}^\sigma} \binom{n}{|F|} = t_\sigma + b_\sigma + i_\sigma + m_\sigma \leq R + m_\sigma - ci_\sigma. \tag{4.8}$$

Summing both sides of (4.8) over all cyclic permutations, we get

$$\sum_\sigma \sum_{F \in \mathcal{F}^\sigma} \binom{n}{|F|} \leq \sum_\sigma (R + m_\sigma - ci_\sigma) = (n-1)!\,R + \sum_{F \in \mathcal{F}_m} (\alpha_F - c\beta_F)\binom{n}{|F|} - \sum_{F \notin \mathcal{F}_m} c\beta_F \binom{n}{|F|}.$$

Now, since for every $F \in \mathcal{F}_m$ we have $\alpha_F - c\beta_F \leq 0$ (by the definition of c), our lemma follows. □

Lemma 22. *If $\mathcal{F}$ contains only sets of size at least 2 and at most $n-2$ and $F \in \mathcal{F}_m$ with $A \subset F \subset B$, then*

$$\frac{\beta_F}{\alpha_F} \geq \frac{|F|\,(n-|F|)}{4} - \frac{n}{2} + 1.$$

Proof. The number of cyclic permutations containing A, F and B is

$$\alpha_F = |A|\,!\,(|F| - |A| + 1)!\,(|B| - |F| + 1)!\,(n - |B|)!\,.$$

The number of cyclic permutations containing only F is (by inclusion/exclusion)

$$\beta_F = |F|\,!\,(n-|F|)! - |A|\,!\,(|F|-|A|+1)!\,(n-|F|)! - |F|\,!\,(|B|-|F|+1)!\,(n-|B|)! + \alpha_F.$$

So we have

$$\begin{aligned}
\frac{\beta_F}{\alpha_F} &= 1 + \frac{|F|\,!\,(n-|F|)!}{|A|\,!\,(|F|-|A|+1)!\,(|B|-|F|+1)!\,(n-|B|)!} \\
&\quad - \frac{(n-|F|)!}{(|B|-|F|+1)!\,(n-|B|)!} - \frac{|F|\,!}{|A|\,!\,(|F|-|A|+1)!} \\
&= \left(\frac{(n-|F|)!}{(|B|-|F|+1)!\,(n-|B|)!} - 1\right)\left(\frac{|F|\,!}{|A|\,!\,(|F|-|A|+1)!} - 1\right) \\
&\geq \min_B \left(\frac{(n-|F|)!}{(|B|-|F|+1)!\,(n-|B|)!} - 1\right) \cdot \min_A \left(\frac{|F|\,!}{|A|\,!\,(|F|-|A|+1)!} - 1\right).
\end{aligned}$$

The first term is minimized by taking $|B| = |F| + 1$, and the second term is minimized by taking $|A| = |F| - 1$. By substituting these values in the inequality above, we get

$$\begin{aligned}
\frac{\beta_F}{\alpha_F} &\geq \left(\frac{n-|F|}{2} - 1\right)\left(\frac{|F|}{2} - 1\right) \\
&= \frac{|F|\,(n-|F|)}{4} - \frac{n}{2} + 1.
\end{aligned}$$

□

Note 4. *If the middle sets in $\mathcal{F}$ all have size at least 3 and at most $n-3$, then for each $F \in \mathcal{F}_m$,*

$$\frac{\beta_F}{\alpha_F} \geq \frac{|F|\,(n-|F|)}{4} - \frac{n}{2} + 1 \geq \frac{n-5}{4}.$$

Therefore,

$$c = \max_{F \in \mathcal{F}_m} \frac{\alpha_F}{\beta_F} \leq \frac{4}{n-5}.$$

Lemma 23. *If $\mathcal{F}$ contains a set of size 1 or $n-1$, then $|\mathcal{F}| < \Sigma_I(n, 2)$ for $n \geq 18$.*

Proof. Assume that $\mathcal{F}$ contains a singleton $\{x\}$. Since $\mathcal{F}$ is intersecting, it follows that $x \in F$ for all $F \in \mathcal{F}$. Then, using the result of Katona and Tarján [51] on families without $A, B \subset C$, we have the bound $|\mathcal{F}| \leq 1 + (1 + 2/n)\binom{n}{\lfloor n/2 \rfloor}$. Note that $\Sigma_I(n, 2) = (2 + o(1))\binom{n}{\lfloor n/2 \rfloor}$, so $\mathcal{F}$ is suboptimal for sufficiently large n ($n \geq 13$ is enough for all our estimates).

Now, assume that $\mathcal{F}$ contains a set of size $n-1$, say S. We define two subfamilies of $\mathcal{F}$. Denote by $\mathcal{F}_1$ the family of those sets in $\mathcal{F}$ which are properly contained in S and set $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Since $\mathcal{F}$ is B-free, it follows that $\mathcal{F}_1$ has no three sets A, B, C with $A, B \subset C$. Thus, the estimate of Katona and Tarján applied to an $n-1$ element ground set yields

$$|\mathcal{F}_1| \leq (1 + \frac{2}{n-1})\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} = (\frac{1}{2} + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Since every set in $\mathcal{F}_2$ contains a fixed element, we can use the result of Katona, De Bonis and Swanepoel [24] applied to an $n-1$ element ground set to show

$$|\mathcal{F}_2| \leq \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor + 1} = (1 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Thus, we get

$$|\mathcal{F}| \leq (\frac{3}{2} + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

which shows that $\mathcal{F}$ will be less than $\Sigma_I(n, 2)$ for n large enough. $\square$

We will now prove some preliminary results we will need about cyclic permutations. We will use the following special case of Lemma 20:

Lemma 24. *Let $\mathcal{G}$ be an intersecting 2-Sperner collection of intervals along a cyclic permutation σ, then*

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq n\Sigma_I(n, 2). \tag{4.9}$$

Equality holds in (4.9) if and only if:

- *n is odd and $\mathcal{G} = \mathcal{H}^{\sigma}_{0,n,2}$;*
- *n is even and $\mathcal{G} = \mathcal{H}^{\sigma}_{x,n,2}$ for some $x \in [n]$.*

We will also need a pair of lemmas that give us an improved bound on 2-Sperner families with a given number of isolated intervals.

Lemma 25. *Let $\mathcal{G}$ be a 2-Sperner family on a cyclic permutation with I isolated intervals, then there are at most $2n - I$ intervals in $\mathcal{G}$.*

Proof. Decompose the cycle into n maximal chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$ as in the proof of Lemma 18. Each isolated interval is found on a different one of the chains. The remaining chains can have at most 2 intervals each. It follows that the total number of intervals is at most $I + 2(n - I) = 2n - I$. $\square$

Lemma 26. *Let $\mathcal{G}$ be a 2-Sperner family of intervals on a cyclic permutation with I isolated intervals, where $1 \leq I \leq n - 1$. Then, there are at most $2n - I - 1$ intervals in total.*

Proof. Consider the canonical cyclic permutation. Since there are at most $n - 1$ isolated intervals, we may take an isolated interval $A_{i^*,j^*} \in \mathcal{C}_{i^*}$ of maximum size such that $\mathcal{C}_{i^*+1}$ contains no isolated interval. We may assume that each $\mathcal{C}$ which contains no isolated interval must contain two intervals from $\mathcal{G}$ (for otherwise the desired upper bound is immediate). The intervals on $\mathcal{C}_{i^*+1}$ have size at least j^* since A_{i^*,j^*} was isolated. Thus, $\mathcal{C}_{i^*+1}$ contains two intervals of size at least j^*, and this, in turn, implies that $\mathcal{C}_{i^*+2}$ has two intervals of size at least j^*. However, eventually we must reach a contradiction since there are only finitely many chains $\mathcal{C}$. $\square$

Now we are ready to prove our main theorem.

Proof of Theorem 25. Let σ be a cyclic permutation. By Lemma 21, it is enough to prove

$$t_\sigma + b_\sigma + (1 + c)i_\sigma \ \leq R. \tag{4.10}$$

If $i_\sigma = 0$, then our family of intervals is 2-Sperner and we are done by Lemma 24. Assume that n is even and $\mathcal{F}^\sigma$ has $I > 0$ isolated intervals. If $I > \frac{n}{2}$, then by Lemma 26, the total number of intervals along σ is less than $\frac{3n}{2} - 1$. Since isolated sets form an antichain, $I \leq n$ by Lemma 18. So the maximum weight of these intervals is at most $\left(\frac{n}{2}\binom{n}{\frac{n}{2}} + \frac{n}{2}\binom{n}{\frac{n}{2}+1}\right)(1 + c) + \left(\frac{n}{2} - 2\right)\binom{n}{\frac{n}{2}+1} < R$, when $n \geq 18$ as desired.

Now, consider the case when there are $2 \leq I \leq n/2$ isolated intervals along σ. By Lemma 26 it follows that the total number of intervals along σ is at most $2n - I - 1$. Pairing off intervals with their complements and considering the maximum weight we can obtain with $2n - I - 1$ intervals, we must show

$$(1+c)I\binom{n}{\frac{n}{2}} + (\frac{n}{2} - I)\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + (\frac{n}{2} - I - 1)\binom{n}{\frac{n}{2}+2} \leq R.$$

Simplifying,

$$cI\binom{n}{\frac{n}{2}} \leq (I-1)\binom{n}{\frac{n}{2}+2}.$$

Dividing through by $\binom{n}{n/2}$, we get

$$I \geq \frac{\frac{n(n-2)}{(n+2)(n+4)}}{\frac{n(n-2)}{(n+2)(n+4)} - c}.$$

By Note 4, we have $c \leq \frac{4}{n-5}$. Substituting this value of c in the above inequality, we get that the right-hand side is strictly less than 2 when n is large enough, as desired. If n is odd, a similar calculation implies that $I > 0$, which settles the odd case completely.

So we may assume that $I = 1$ (n is even) and that the total number of intervals along σ is exactly $2n-2$ (If we have less than $2n-2$ intervals and $I = 1$, it can be checked easily that $t_\sigma + b_\sigma + (1+c)i_\sigma < R$ for large enough n). Now, the intervals in $\mathcal{T}_\sigma \cup \mathcal{B}_\sigma \cup \mathcal{I}_\sigma$ form a 2-Sperner family of intervals along σ. Let us call the subfamily of maximal intervals (i.e., those intervals that are not contained in any other interval) $\mathcal{U}$ and the subfamily of minimal intervals (i.e., those intervals that do not contain any other interval) $\mathcal{D}$. Now, if either $\mathcal{U}$ or $\mathcal{D}$ contains n intervals, then, since it is an intersecting antichain, it has to be a complete level. In this case, by a similar argument as in the proof of Lemma 24 we have $t_\sigma + b_\sigma + (1+c)i_\sigma < R$ for large enough n. So we can assume that both $\mathcal{U}$ and $\mathcal{D}$ contain at most $n-1$ intervals. Since the interval in $\mathcal{I}_\sigma$ is both maximal and minimal we have $|\mathcal{U} \cap \mathcal{D}| \geq 1$. But then, the total number of intervals in our 2-Sperner family is $|\mathcal{U} \cup \mathcal{D}| = |\mathcal{U}| + |\mathcal{D}| - |\mathcal{U} \cap \mathcal{D}| \leq 2n-3$, a contradiction.

We now establish the cases of equality. First let us notice that by Lemma 21, we have $|\mathcal{F}| = \frac{R}{n}$ if and only if we have equality in (4.10) for each σ. However, we just saw that if $I > 0$, the inequality (4.10) is never sharp when n is large enough (for both the n is even case and n is odd case). Thus, we have $I = 0$ for every σ. However, since any middle set of $\mathcal{F}$ appears as an isolated interval on some σ, we may conclude that $\mathcal{F}$ has no middle sets (i.e., $|\mathcal{F}_m| = 0$). Therefore, $\mathcal{F}$ is 2-Sperner and so the equality cases follow from Theorem 29. □

4.3 Bollobás and Greene-Katona-Kleitman-type inequalities

In this subsection we will prove Theorem 27. The proof of Theorem 26 uses the exact same idea but with the weight function defined in [35] to prove Theorem 9.

Proof. Following [12] we will use the weight function

$$f(A,\sigma) = \begin{cases} \frac{1}{|F|}, & \text{if } F \in \mathcal{F} \text{ and } F \text{ is an interval in } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

On the one hand, we have

$$\sum_{F\in\mathcal{F}} \sum_{\sigma} w(F,\sigma) = \sum_{F\in\mathcal{F}} (|F|-1)!\,(n-|F|)!\,.$$

We will show

$$\sum_{\sigma} \sum_{F\in\mathcal{F}} w(F,\sigma) \le 2(n-1)!\,.$$

Fix a cyclic permutation σ. As before, let $\mathcal{I}_\sigma$ be the collection of middle intervals along σ. Similarly, let $\mathcal{M}_\sigma$ be the collection of isolated intervals along σ. Then, the following inequality holds:

$$w(\mathcal{F}^\sigma) \le 2 - w(\mathcal{I}_\sigma) + w(\mathcal{M}_\sigma). \tag{4.11}$$

Indeed, initially leave out all sets in $\mathcal{M}_\sigma$ and $\mathcal{I}_\sigma$. The remaining sets may be partitioned into two antichains along σ, say $\mathcal{A}_1$ and $\mathcal{A}_2$. Clearly $\mathcal{A}_1 \cup \mathcal{I}_\sigma$ is an antichain, as is $\mathcal{A}_2 \cup \mathcal{I}_\sigma$. By the argument from [12] we have $w(\mathcal{A}_1 \cup \mathcal{I}_\sigma) \le 1$ and $w(\mathcal{A}_2 \cup \mathcal{I}_\sigma) \le 1$. Thus, summing we have

$$w(\mathcal{A}_1) + w(\mathcal{A}_2) + 2w(\mathcal{I}_\sigma) \le 2.$$

Rearranging and adding $w(\mathcal{M}_\sigma)$ to both sides yields (4.11).

Since the only possible middle sets along a cyclic permutation are middle sets in $\mathcal{F}$ (that is, $\mathcal{M}_\sigma \subseteq \mathcal{F}_m^\sigma$), summing up we have

$$\sum_{\sigma} \sum_{F\in\mathcal{F}} w(F,\sigma) \le 2(n-1)! + \sum_{M\in\mathcal{F}_m} \frac{\alpha_M}{|M|} - \sum_{M\in\mathcal{F}_m} \frac{\beta_M}{|M|}.$$

We have seen already that $\beta_M \geq \alpha_M$, and the proof is complete. $\square$

4.4 Results for general posets P

In this subsection we prove Theorem 28. Before we start the proof we recall the notion of a *double chain* introduced in [14] (and discussed in the introduction).

Definition 2 (Double chain)**.** *Let $\emptyset = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = [n]$ be a maximal chain (so $|A_i| = i$). The double chain associated to this chain is given by $\mathcal{D} = \{A_0, A_1, \ldots, A_n, M_1, M_2, \ldots, M_{n-1}\}$, where $M_i = A_{i-1} \cup \{A_{i+1} \setminus A_i\}$.*

We will now introduce the notion of a *double chain-complement pair* which is the key ingredient of the proof.

Definition 3 (Double chain-complement pair)**.** *Let $\mathcal{D}$ be a double chain. By complementing all the sets in $\mathcal{D}$ we get another double chain $\mathcal{D}'$. We refer to $\mathcal{H} = \mathcal{D} \cup \mathcal{D}'$ as a double chain-complement pair.*

In the rest of this subsection we shall work with the double chain-complement pair $\mathcal{H}_0 = \mathcal{D}_0 \cup \mathcal{D}_0'$ where $\mathcal{D}_0$ is defined by taking $A_i = [i]$; other double chain-complement pairs are related to it by permutation. Let $\pi \in S_n$ be a permutation and $F \subseteq [n]$ be a set, then F^π denotes the set $\{\pi(a) : a \in F\}$. We define the double chain-complement pair $\mathcal{H}_0^\pi$ to be the collection $\{F^\pi : F \in \mathcal{H}_0\}$. Notice that this gives us $n!$ double chain-complement pairs in total.

Now we are ready to prove our theorem. Let $\mathcal{F}$ be an intersecting P-free family. We will use the collections $\mathcal{H}_0^\pi = \mathcal{D} \cup \mathcal{D}'$ for a weighted double counting argument described below.

Define a weight function $f(F, \mathcal{H}_0^\pi)$ by

$$f(F, \mathcal{H}_0^\pi) = \begin{cases} \binom{n}{|F|}, & \text{if } F \in \mathcal{F},\ F \neq [n] \text{ and } F \in \mathcal{H}_0^\pi \\ 4, & \text{if } F \in \mathcal{F},\ F = [n] \\ 0, & \text{otherwise.} \end{cases}$$

We want to compute $\sum_F \sum_{\mathcal{H}_0^\pi} f(F, \mathcal{H}_0^\pi)$ in two different ways. First let us fix a $F \in \mathcal{F}$ and determine how many collections $\mathcal{H}_0^\pi$ contain F. If $F = [n]$ we know that all $n!$ collections $\mathcal{H}_0^\pi$ contain it. So let us assume $F \neq [n]$. Let H_1, H_2, H_3, H_4 be the four sets in $\mathcal{H}_0$ of size $|F|$. The number of permutations π such that a given H_i (where $1 \le i \le 4$) is mapped to F is $|F|\,!\,(n-|F|)!$, since we can map the elements of H_i to F arbitrarily and the elements of $[n] \setminus H_i$ to $[n] \setminus F$ arbitrarily. So it follows that the number of permutations π such that $F \in \mathcal{H}_0^\pi$ is $4\,|F|\,!\,(n-|F|)!$. Thus, we have

$$\sum_F \sum_{\mathcal{H}_0^\pi} f(F, \mathcal{H}_0^\pi) = 4\,|\mathcal{F}|\,n!\,. \tag{4.12}$$

Now let us fix a $\mathcal{H}_0^\pi$. Since n is odd, there are 8 sets in $\mathcal{H}_0^\pi$ of maximal weight $\binom{n}{\lfloor n/2 \rfloor}$ and 8 sets of second largest weight $\binom{n}{\lfloor n/2 \rfloor+1}$ and so on. The 8 sets of $\mathcal{H}_0^\pi$ of the same weight $\binom{n}{\lfloor n/2 \rfloor+i}$ (where $i \ge 1$) consist of 4 sets and their respective complements. Thus, at most 4 of them can belong to our family $\mathcal{F}$ (because $\mathcal{F}$ is intersecting). Now let us recall a lemma due to Burcsi and Nagy [14].

Lemma 27 (Burcsi-Nagy [14])**.** *Let P be a poset. Any subset of size $|P| + h(P) - 1$ of a double chain contains P as a subposet.*

Since a P-free family has at most $|P| + h(P) - 2$ sets on a double chain, it follows that we can have at most $2(|P| + h(P) - 2)$ sets in $\mathcal{F} \cap \mathcal{H}_0^\pi$ for any π. Since we can have at most 4 sets of weight $\binom{n}{\lfloor n/2 \rfloor+i}$ in $\mathcal{F} \cap \mathcal{H}_0^\pi$, the total weight of sets in $\mathcal{F} \cap \mathcal{H}_0^\pi$ is at most $\sum_{i=1}^{\frac{2(|P|+h(P)-2)}{4}} 4\binom{n}{\lfloor n/2 \rfloor+i}$. So we have

$$\sum_{\mathcal{H}_0^\pi} \sum_F f(F, \mathcal{H}_0^\pi) \le n! \left(\sum_{i=1}^{\frac{|P|+h(P)}{2}-1} 4\binom{n}{\lfloor \frac{n}{2} \rfloor + i} \right). \tag{4.13}$$

Combining (4.12) and (4.13), we have the desired bound,

$$|\mathcal{F}| \le \sum_{i=1}^{\frac{|P|+h(P)}{2}-1} \binom{n}{\lfloor \frac{n}{2} \rfloor + i}.$$

Chapter 5

A De Bruijn-Erdős theorem for posets

The starting point of this chapter is a classical result of De Bruijn and Erdős in combinatorial geometry. A set of n points in the plane, not all on a line, is called a *near-pencil* if exactly $n-1$ of the points are collinear.

Theorem 30 (De Bruijn, Erdős [25])**.** *Every noncollinear set of n points in the plane determines at least n lines. Moreover, equality occurs if and only if the configuration is a near-pencil.*

Erdős [27] showed that this result is a consequence of the Sylvester-Gallai theorem which asserts that every noncollinear set of n points in the plane determines a line containing precisely two points. Later, De Bruijn and Erdős [25] proved a more general combinatorial result which implies Theorem 30.

In an arbitrary metric space (V, d), Menger [63] defined the following natural notion of *betweenness* on V:

$$[axb] \Leftrightarrow d(a,x) + d(x,b) = d(a,b).$$

Using this definition of betweenness, one can also give a simple abstract definition of a *line*:

$$\overline{ab} = \{a,b\} \cup \{x : [xab] \text{ or } [axb] \text{ or } [abx]\}. \tag{5.1}$$

Using (5.1), the line $\overline{ab}$ is defined for any two distinct points a and b in V. Observe that this definition of a line generalizes the classical notion of a line in Euclidean space to any metric space. These lines may have strange properties: two lines might have more than one common point, and it is even possible for a line to be a proper subset of another line. A line is called *universal* if it contains all points in V. Chen and Chvátal [17] proposed the following conjecture, which, if true, would give a vast generalization of Theorem 30.

Conjecture 1 (Chen-Chvátal [17]). *Any finite metric space on n points either induces at least n distinct lines or contains a universal line.*

Although the conjecture has been proved in several special cases [3, 8, 18, 20, 19, 42], it is still wide open. The best known lower bound on the number of lines in a general finite metric space with no universal line is $(1/\sqrt{2}+o(1))\sqrt{n}$ [2].

Recently, Chen and Chvátal [17] generalized the notion of lines in metric spaces to lines in hypergraphs. Recall that a *hypergraph* is an ordered pair $(V, \mathcal{E})$ such that V is a set of elements called the *vertices* and $\mathcal{E}$ is a family of subsets of V called the *edges*. Also recall that a hypergraph is *k-uniform* if each of its edges consist of k vertices. They observed that given a metric space (V, d), one can associate a hypergraph $H(d) = (V, \mathcal{E})$ with $\mathcal{E} := \{\{a, b, c\} : [abc] \text{ in } (V, d)\}$. If the line $\overline{ab}$ in the 3-uniform hypergraph is defined as

$$\overline{ab} = \{a, b\} \cup \{x : \{a, b, x\} \in \mathcal{E}\},$$

then the metric space (V, d) and the hypergraph $(V, \mathcal{E})$ determine the same set of lines.

They proved that there is an infinite family of 3-uniform hypergraphs inducing only $c^{\sqrt{\log_2 n}}$ distinct lines (where n is the number of vertices and c is a constant). This means that there are infinitely many 3-uniform hypergraphs for which the analogue of Theorem 30 does not hold. However, analogues of Theorem 30 have been shown to hold for some special families of 3-uniform hypergraphs in [9]. The best known lower bound on the number of lines in a 3-uniform hypergraph with no universal line is $(2 - o(1)) \log_2 n$ [1].

Following the lead of these previous works, we obtain an analogue of De Bruijn-Erdős theorem for posets. Let $P = (X, \prec)$ be a finite poset with the order relation $\prec$ defined on the set X. Recall that the size of a maximum chain in P is called the *height* of P and is denoted $h(P)$.

As in the metric space case, a poset P induces a natural betweenness relation:

$$[abc] \Leftrightarrow a \prec b \prec c \text{ or } c \prec b \prec a.$$

Therefore, we can again define lines in posets using (5.1). Observe that, if a is incomparable to b, then the line $\overline{ab} = \{a, b\}$ and, if a is comparable to b, then

$$\overline{ab} = \{a, b\} \cup \{x : x \text{ is comparable to both } a \text{ and } b\}.$$

As before, a line is *universal* if it contains every point from the ground set. Our main result is to show that an analogue of Conjecture 1 holds for posets. In fact, we obtain a stronger bound as a function of the height of the poset.

Theorem 31 (Aboulker, Lagarde, Malec, Methuku, T [4])**.** *If P is a poset on n vertices with no universal line, and $h(P) \geq 2$, then P induces at least*

$$h(P)\binom{\lfloor n/h(P)\rfloor}{2} + \lfloor n/h(P)\rfloor(n \bmod h(P)) + h(P) \tag{5.2}$$

distinct lines with equality if $h(P) \geq n/2$.

Observe that (5.2) is always greater than or equal to n (with equality if $h(P) \geq \lfloor n/2 \rfloor$). Moreover, if $h(P) = O(n^s)$ for $0 < s \leq 1$, the number of distinct lines in P is $\Omega(n^{2-s})$.

Our second result is a generalization from posets to graphs. For any graph $G = (V, E)$ and vertices $a, b \in V$, we can define the line $\overline{ab}$ as

$$\overline{ab} = \{a, b\} \cup \{c : abc \text{ is a triangle}\}.$$

Again, the line $\overline{ab}$ is universal if it contains every vertex in V. We prove

Theorem 32 (Aboulker, Lagarde, Malec, Methuku, T [4])**.** *If a graph G on $n \geq 4$ vertices does not contain a universal line, then it induces at least n distinct lines, and equality occurs only if G consists of a clique of size $n-1$ and a vertex that has at most one neighbor in the clique.*

Remark 1. *It may be easily seen that the theorem also holds when $n = 3$, but we have an additional extremal example in this case: a graph where all pairs of vertices are non-adjacent.*

Observe that to any poset $P = (V, \prec)$, we can associate a graph $G = (V, E)$ where $ab \in E$ if and only if $a \prec b$ or $b \prec a$. Such a graph is called a *comparability graph*. Hence, for any three vertices

a, b, c of P, we have a, b and c all comparable to each other if and only if abc is a triangle in the corresponding comparability graph. Therefore, the graph case is a generalization of the poset case.

Using Theorem 32, it can be easily seen that, in the case of posets, equality occurs only if the poset consists of a chain of size $n-1$ and a vertex which is comparable to at most one vertex of this chain.

This chapter is organized as follows. In Subsection 5.1, we prove Theorem 31 by providing an algorithm for finding lines. In Subsection 5.2, we prove Theorem 32 by induction.

5.1 Lines in posets

We begin by introducing some notation that will be useful in the proof of Theorem 31. For any pair of elements a, b in a poset $P = (X, \prec)$, we write $a \nsim b$ to indicate that the points a and b are not comparable (that is, neither $a \prec b$ nor $b \prec a$ hold). Let $Y \subseteq X$. We denote by $P \setminus Y$ the poset on the set of points $X \setminus Y$ together with $\prec$ restricted to $X \setminus Y$.

In this subsection, we prove a lower bound on the number of lines in a poset as a function of its height. Before we proceed with the proof, we need a simple lemma.

Lemma 28. *If $A_1, ..., A_r$ are r sets such that $\sum_{i=1}^{r} |A_i| = n$, then $\sum_{i=1}^{r} \binom{|A_i|}{2} \geq r\binom{\lfloor n/r \rfloor}{2} + \lfloor n/r \rfloor (n \bmod r)$.*

Proof. Observe first that if $||A_i| - |A_j|| \leq 1$ for all i and j, then the bound holds. Thus, it suffices to prove that if $|A_i| - |A_j| > 1$, then moving one point from A_i to A_j does not increase $\sum_{i=1}^{r} \binom{|A_i|}{2}$. Let $x \in A_i \setminus A_j$ and $A_i' = A_i \setminus \{x\}$, $A_j' = A_j \cup \{x\}$. We have

$$\binom{|A_i|}{2} + \binom{|A_j|}{2} \geq \binom{|A_i| - 1}{2} + \binom{|A_j| + 1}{2} = \binom{|A_i'|}{2} + \binom{|A_j'|}{2}$$

by the convexity of $\binom{m}{k}$ in m, and the lemma follows. □

5.1.1 Proof of Theorem 31

Let $\mathcal{A}$ be a minimal partition of P into antichains, and let $C \subseteq P$ be a maximal chain in P. By Mirsky's theorem, we know that $|\mathcal{A}| = |C| = h(P)$. For notational convenience, from now on, let $h(P)$ be denoted by H. Denote the elements of $\mathcal{A}$ and C, respectively, as $\mathcal{A} = \{A_1, \ldots, A_H\}$

and $C = \{c_1 \dots c_H\}$ with $c_1 \prec \dots \prec c_H$. Assume, without loss of generality, that $c_i \in A_i$ for $i = 1, \dots, H$.

Set

$$\mathcal{L}_0 := \bigcup_{i=1}^{H} \{\overline{ab} : a, b \in A_i, a \neq b\}.$$

Note that all of the lines in $\mathcal{L}_0$ are induced by incomparable points and are, thus, pairwise distinct. By Lemma 28, we have

$$|\mathcal{L}_0| = \sum_{i=1}^{H} \binom{|A_i|}{2} \geq H\binom{\lfloor n/H)\rfloor}{2} + \lfloor n/H\rfloor(n \bmod H).$$

Next we use the chain C to find H further lines, distinct from those in $\mathcal{L}_0$. We do so via the following iterative process:

Set $b_1 = 1$, $t_1 = H$ and $\mathcal{L}_1 = \emptyset$. For $k = 1, 2, \dots$, apply the following steps until a STOP condition is met.

Step 1 If $b_k = t_k$, set $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_1 c_H}\}$ and **STOP**.

Otherwise $b_k < t_k$ and there exists $s_k \notin \overline{c_{b_k} c_{t_k}}$. If s_k is incomparable with both c_{b_k} and c_{t_k}, go to **Step 2a**. If s_k is incomparable with c_{b_k} and comparable with c_{t_k}, go to **Step 2b**. Finally, if s_k is incomparable with c_{t_k} and comparable with c_{b_k}, go to **Step 2c**.

Step 2a Set $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : b_k \leq i \leq t_k\} \cup \{\overline{c_1 c_H}\}$ and **STOP**.

Step 2b Set $b_{k+1} = \min_{b_k \leq i < t_k}\{i : c_i \sim s_k\}$, $t_{k+1} = t_k$, $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : b_k \leq i \leq b_{k+1}\}$. Go to **Step 1**.

Observe that, in this case, we have that $b_k < b_{k+1} \leq t_k$, $s_k \prec c_{b_{k+1}}$ and, for $j = b_k, \dots, b_{k+1}-1$, we have $s_k \nsim c_j$.

Step 2c Set $t_{k+1} = \min_{b_k < i \leq t_k}\{i : c_i \sim s_k\}$,

$b_{k+1} = b_k$, $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : t_{k+1} \leq i \leq t_k\}$. Go to **Step 1**.

Observe that, in this case, we have that $b_k \leq t_{k+1} < t_k$, $c_{t_{k+1}} \prec s_k$ and, for $j = t_{k+1}+1, \dots, t_k$, $s_k \nsim c_j$.

Assume that the process stops after K iterations.

For any $k < K$, in the k^{th} iteration, exactly one line added to $\mathcal{L}_k$ is induced by two comparable points. We call this line l_k. Thus, there are $K-1$ such lines, namely $l_1, \dots, l_{K-1}$. Notice that l_k is either $\overline{c_{b_{k+1}} s_k}$ or $\overline{c_{t_{k+1}} s_k}$.

If $l_k = \overline{c_{b_{k+1}} s_k}$, since $s_k \prec c_{b_{k+1}}$, we have $\{c_{b_{k+1}}, c_{b_{k+2}}, \dots c_{b_K}, c_{t_K}, c_{t_{K-1}}, \dots, c_{t_1}\} \subseteq \overline{c_{b_{k+1}} s_k}$, and since $c_{b_k} \not\sim s_k$, we have $c_{b_k} \notin \overline{c_{b_{k+1}} s_k}$. Similarly, if $l_k = \overline{c_{t_{k+1}} s_k}$ we have $\{c_{b_1}, \dots, c_{b_K}, c_{t_K}, \dots, c_{t_{k+1}}\} \subseteq \overline{c_{t_{k+1}} s_k}$ and $c_{t_k} \notin \overline{c_{t_{k+1}} s_k}$. Observe now that the line $\overline{c_1 c_H}$, that is added at the K^{th} iteration, contains all points in C. This implies that the lines $l_1, \dots, l_{K-1}, \overline{c_1 c_H}$ are pairwise distinct. Thus, the process finds K pairwise distinct lines which are induced by comparable points. Moreover, since all the lines in $\mathcal{L}_0$ are induced by incomparable points, none of these K lines belong to $\mathcal{L}_0$.

The rest of the lines found by the process are induced by incomparable points. Hence, it remains to prove that $H-K$ of them are pairwise distinct and don't belong to $\mathcal{L}_0$.

Let $k < K$. We claim that $\mathcal{L}_k$ contains at least $b_{k+1} - b_k + t_k - t_{k+1} - 1$ (new) lines that are not in $\mathcal{L}_{k-1}$. Assume first that, in the k^{th} iteration, lines are added at Step 2b (so $t_k - t_{k+1} = 0$). So $b_{k+1} - b_k$ lines induced by two incomparable points are added, namely $\overline{c_{b_k} s_k}, \dots, \overline{c_{b_{k+1}-1} s_k}$. At most one of these lines belongs to $\mathcal{L}_0$ and none of them belongs to $\mathcal{L}_{k-1} \setminus \mathcal{L}_0$ because lines induced by incomparable points that are added in previous iterations of the process, involve points of C either strictly below b_k or strictly above $t_k \geq b_{k+1}$. Hence, at least $b_{k+1} - b_k - 1$ new lines induced by incomparable points are added at Step 2b. A symmetric argument proves that, in the case where the lines are added at Step 2c (so $b_{k+1} - b_k = 0$), we have added $t_k - t_{k+1} - 1$ new lines induced by incomparable points.

So, after $K-1$ iterations, the number of lines induced by incomparable points, in $\mathcal{L}_{K-1} \setminus \mathcal{L}_0$ is

$$\sum_{k=1}^{K-1} (b_{k+1} - b_k + t_k - t_{k+1} - 1) = t_1 - b_1 - (t_K - b_K) - (K-1) = H - K - (t_K - b_K).$$

Hence, it remains to show that $t_K - b_K$ new distinct lines induced by incomparable points are added at the K^{th} iteration. In the case where $b_K = t_K$ we are done, so we may assume that $b_K < t_K$ and the process terminates at Step 2a. So, the lines $\overline{c_{b_K} s_k}$, $\overline{c_{b_K+1} s_k}, \dots$, $\overline{c_{t_K} s_k}$ are added. At most one of these lines belongs to $\mathcal{L}_0$ and none of them belongs to $\mathcal{L}_{K-1} \setminus \mathcal{L}_0$ since lines induced by incomparable points added at iterations $1, \dots, K-1$ involve points of C either strictly below b_K or strictly above t_K. It follows that $t_K - b_K$ new lines are added.

5.2 Lines in graphs

We first need two easy observations about lines in a graph $G = (V, E)$. A vertex x in a graph is *universal* if it is adjacent to all vertices in $V \setminus x$.

1. If $ab \notin E$, then $\overline{ab} = \{a, b\}$,

2. A line $\overline{ab}$ is universal if and only if both a and b are universal.

We are now ready to prove our generalization to the graph case. We will use induction on n on the full statement of the theorem. First, we show that the theorem holds when $n = 4$. If there is no triangle in our graph, then every pair of vertices induces a distinct line, giving us 6 lines. If there are two different triangles in our graph, then there exist 2 vertices p, q, that belong to both triangles, and the line $\overline{pq}$ is universal, a contradiction. Therefore, we have exactly one triangle, and it is easy to see that in this case we have exactly 4 lines in our graph and the extremal graphs are exactly as desired.

Let $G = (V, E)$ be a graph on $n \geq 5$ vertices having no universal lines, and assume the statement holds for smaller n.

Let $V_1 \subseteq V$ be the set of those points x such that $G \setminus \{x\}$ has a universal line, and set $V_2 = V \setminus V_1$. Assume first that $V_2 = \emptyset$. So $V_1 = V$ and, thus, for any $x \in V$, $V \setminus \{x\}$, induces a universal line. Since G has no universal lines, $V \setminus \{x\}$ is a line of G for any $x \in V$. Thus, G induces n distinct lines of size $n - 1$. Moreover, since it has no universal lines, G has at least two non-adjacent vertices, providing a line of size two. Thus, if $V_2 = \emptyset$, we have that G induces at least $n + 1$ lines.

So we may assume from now on that $V_2 \neq \emptyset$. We will distinguish between two cases:

Case 1. There exists a point x in V_2 that is not universal.

Let y be a vertex non-adjacent to x. Since $G \setminus \{x\}$ has no universal lines, by induction, $G \setminus \{x\}$ induces at least $n - 1$ distinct lines. If ℓ is a line of $G \setminus \{x\}$, then either ℓ or $\ell \cup \{x\}$ is a line of G. It follows that these lines are all distinct in G. Moreover, if they contain x, then they have at least 3 vertices and so they are all distinct from $\overline{xy} = \{x, y\}$. Hence, G has at least n distinct lines.

Now, assume that G induces exactly n distinct lines. Then, $G \setminus \{x\}$ must contain exactly $n - 1$ lines and so by induction, $G \setminus \{x\}$ consists of a clique K on $n - 2$ vertices, $x_1, x_2, \ldots, x_{n-2}$, and a vertex z which has at most one neighbor in K. Notice that the set of lines of $G \setminus \{x\}$

is $\mathcal{L}_{G\setminus\{x\}} := \{\{x_1, \ldots, x_{n-2}\}, \{z, x_1\}, \ldots, \{z, x_{n-2}\}\}$, giving us $n-1$ distinct lines of G, namely, $\mathcal{L} := \{\ell \text{ or } \ell \cup \{x\} \mid \ell \in \mathcal{L}_{G\setminus\{x\}}\}$.

We claim that x is adjacent to all vertices of $V \setminus \{x, y\}$ because otherwise there exists a vertex y' in $V \setminus \{x, y\}$ such that $\overline{xy'} = \{x, y'\}$ is a line of G. Since $\overline{xy} = \{x, y\}$ is also a line of G, and $\overline{xy}, \overline{xy'} \notin \mathcal{L}$, G induces at least $n+1$ distinct lines contradicting our assumption.

Assume that $y \in K$, and let z' be the unique neighbor of z in K. Consider a vertex w in $K \setminus \{y, z'\}$ (such a vertex exists because $n \geq 5$). Since $y \notin \overline{xw}$ and $z \notin \overline{xw}$, we have $\overline{xw} \notin \mathcal{L}$. Of course, $\overline{xy} \notin \mathcal{L}$ is a line of G like before. Thus, G induces at least $n+1$ distinct lines again. Therefore, $y = z$, and $K \cup \{x\}$ is a clique of G as desired.

Case 2. All points of V_2 are universal.

Since G has no universal line, if follows that V_2 contains exactly one vertex, say x. So $V_1 = V \setminus \{x\}$ and, thus, for any $u \in V_1$, $V \setminus \{u\}$, is a line of G. This yields $n-1$ lines of size $n-1$. Moreover, it is easy to see that, since G has no universal lines, it must contain at least two pairs of non-adjacent vertices, providing us with two more distinct lines of size two. Hence, G has at least $n+1$ distinct lines.

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